Section 3.6 **Variation of Parameters**

Consider the nonhomogeneous linear second order differential equation

$$y'' + p(x)y' + q(x)y = g(x).$$

Let $\{y_1(x), y_2(x)\}$ be a fundamental solution set to the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

The general solution to this homogeneous equation is $y_h(x) = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are constants. To find a particular solution of the nonhomogeneous equation we assume that $c_1 = c_1(x)$ and $c_2 = c_2(x)$ are functions of x and we seek a particular solution $y_p(x)$ in the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

Let's substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into (1):

We can find $c_1(x)$ and $c_2(x)$ solving the system

$$\begin{cases} c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = g(x) \end{cases}$$

for $c_1'(x)$ and $c_2'(x)$.

$$\boxed{c_1(x) = \int \frac{-g(x)y_2(x)}{W[y_1, y_2](x)}dx}, \quad \boxed{c_2(x) = \int \frac{g(x)y_1(x)}{W[y_1, y_2](x)}dx}$$

Example. Find the general solution to the equation $y'' + y = \frac{1}{\sin x}$.

homogeneous eqn.

$$y'' + y = 0$$

auxiliary eqn. $r^2 + 1 = 0$

$r = \pm i$

general solution of the homogeneous eqn

$$y_h(x) = c_1 \cos x + c_2 \sin x$$

General solution of the nonhomogeneous eqn.

$$y(x) = C_1(x)\underbrace{\cos x}_{y_1(x)} + C_2(x)\underbrace{\sin x}_{y_2(x)}$$

$$y'(x) =$$

$$\begin{cases} C_1'(x)\underbrace{\cos x}_{y_1(x)} + C_2'\underbrace{\sin x}_{y_2(x)} = 0 \Rightarrow C_2' = -\dfrac{C_1' \cos x}{\sin x} \\ C_1'\underbrace{(-\sin x)}_{y_1'(x)} + C_2'\underbrace{\cos x}_{y_2'(x)} = \dfrac{1}{\sin x} \end{cases}$$

$$-\sin x \, C_1' - \frac{C_1' \cos^2 x}{\sin x} = \frac{1}{\sin x}$$

$$-\frac{C_1'\overbrace{(\sin^2 x + \cos^2 x)}^{1}}{\sin x} = \frac{1}{\sin x}$$

$$C_1' = -1 \Rightarrow \boxed{C_1 = -x + C_3}$$

$$C_2' = +\frac{\cos x}{\sin x} \Rightarrow C_2 = \int \frac{\cos x}{\sin x}\,dx = \boxed{\ln|\sin x| + C_4 = C_2}$$

$$\boxed{y(x) = (-x + C_3)\cos x + (\ln|\sin x| + C_4)\sin x}$$