Section 3.7 **Mechanical and electrical vibrations**

A damped mass-spring oscillator consists of a mass m attached to a spring fixed at one end. Model for the motion of the mass is expressed by the initial value problem

$$my'' + by' + ky = F_{\text{external}}, \quad y(0) = y_0, \ \ y'(0) = v_0,$$

where m is a mass, b is the damping coefficient, k is the stiffness.

Let's $F_{\text{external}} = 0$

Undamped free case: $b = 0$
The equation reduces to

$$my'' + ky = 0| : m$$

$$y'' + \omega^2 y = 0$$

where $\omega = \sqrt{\frac{k}{m}}$. The solution of this equation is

$$y(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

$$y(t) = A \sin(\omega t + \phi),$$

where $A = \sqrt{C_1^2 + C_2^2}$, $\tan \phi = \frac{C_1}{C_2}$.

The motion is periodic with
period $2\pi/\omega$
natural frequency $\omega/2\pi$
angular frequency ω
amplitude A.

Example 1. A 2-kg mass is attached to a spring with stiffness $k = 50$ N/m. The mass is displaced 1/4 m to the left of the equilibrium point and given a velocity of 1 m/sec to the left. Neglecting damping,

a) Set up an initial value problem for this system.

b) Find the equation of motion of the mass.

c) Find the amplitude, period and frequency of the motion.

Underdamped or oscillatory motion ($b^2 < 4mk$)
The solution to the equation

$$my'' + by' + ky = 0$$

is

$$y(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) = Ae^{\alpha t} \sin(\beta t + \phi),$$

where
$\alpha = -\dfrac{b}{2m}$, $\beta = \dfrac{1}{2m}\sqrt{4mk - b^2}$, $A = \sqrt{C_1^2 + C_2^2}$, $\tan \phi = \dfrac{C_1}{C_2}$.

The solution $y(t)$ varies between $-Ae^{\alpha t}$ and $Ae^{\alpha t}$ with **quasiperiod** $P = \dfrac{2\pi}{\beta} = \dfrac{4\pi m}{\sqrt{4mk - b^2}}$ and **quasifrequency** $1/P$. $y(t) \to 0$ as $t \to \infty$.

An exponential factor $Ae^{\alpha t}$ is called a **damping factor**.

The system is called **underdamped** because there is not enough damping present (b is too small) to prevent the system from oscillating.

Overdamped motion ($b^2 > 4mk$)
The solution to the equation

$$my'' + by' + ky = 0$$

is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

where $r_1 = -\dfrac{b}{2m} + \dfrac{1}{2m}\sqrt{4mk - b^2}$, $r_2 = -\dfrac{b}{2m} - \dfrac{1}{2m}\sqrt{4mk - b^2}$
$r_2 < 0$, and since $b^2 > b^2 - 4mk$, $r_1 < 0$. $y(t) \to 0$ as $t \to \infty$.

Critically damped motion ($b^2 = 4mk$) The solution to the equation

$$my'' + by' + ky = 0$$

is

$$y(t) = (c_1 + c_2 t)e^{-\frac{b}{2m}t}.$$

$y(t) \to 0$ as $t \to \infty$.

Example 2. The motion of the mass-spring system with damping is governed by

$$y''(t) + by'(t) + 64y(t) = 0, \quad y(0) = 1, \;\; y'(0) = 0$$

Find the equation of the motion for $b = 10, 16, 20$.

Electric circuits.

If Q is the charge at time t in an electrical closed circuit with inductance L, resistance R, and capacitance C, then by Kirchhoff's Second Law (from Physics) the impressed voltage $E(t)$ is equal to the sum of the voltage drops in the rest of the circuit

$$E(t) = IR + \frac{Q}{C} + LI'(t)$$

By substitution $I = Q'$ we get

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

Analogy between electrical and mechanical quantities:

Charge Q	Position u
Inductance L	mass m
Resistance R	Damping constant γ
Inverse capacitance $1/C$	Spring constant k
Impressed voltage $E(t)$ (electromotive force)	External force $F(t)$