Section 6.1 **Definition of the Laplace Transform.**

Definition 1. Let $f(t)$ be a function on $[0, \infty)$. The Laplace transform of f is the function F defined by the integral

$$\mathcal{L}\{f\}(s) = \boxed{F(s) = \int_0^\infty f(t)e^{-st}dt.}$$

The domain of $F(s)$ is all the values of s for which integral exists. The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Notice, that integral in definition is **improper** integral.

$$\int_0^\infty f(t)e^{-st}dt = \lim_{N\to\infty} \int_0^N f(t)e^{-st}dt$$

whenever the limit exists.

Example 1. Determine the Laplace transform of the given function.

1. $f(t) = 1$, $t \geq 0$.

$$\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}dt = \lim_{N\to\infty} \int_0^N e^{-st}dt = -\frac{1}{s} \lim_{N\to\infty} \left(e^{-st}\right)_{t=0}^{t=N}$$

$$= -\frac{1}{s}\left[\cancel{\lim_{N\to\infty} e^{-Ns}}^{\,0} - 1\right] = \boxed{\frac{1}{s}}$$

2. $f(t) = t$, $t \geq 0$.

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}dt = \lim_{N\to\infty} \int_0^N te^{-st}dt$$

by parts: $\int u\,dv = uv - \int v\,du$

$u = t$ $\quad dv = e^{-st}dt$

$du = dt$ $\quad v = -\frac{1}{s}e^{-st}$

$$= \lim_{N\to\infty}\left[-\frac{t}{s}e^{-st}\Big]_{t=0}^{t=N} + \frac{1}{s}\int_0^N e^{-st}dt\right]$$

$$= \lim_{N\to\infty}\left[-\frac{N}{s}e^{-Ns} - \frac{1}{s^2}e^{-st}\Big]_{t=0}^{t=N}\right] = \lim_{N\to\infty}\left[-\frac{N}{s}\cancel{e^{-Ns}}^{\,0} - \frac{1}{s^2}\cancel{e^{-Ns}}^{\,0} + \frac{1}{s^2}\right]$$

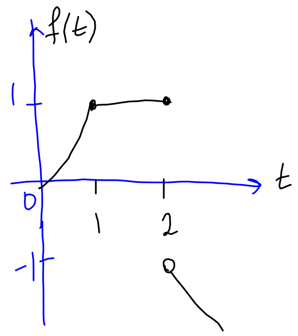
$$= \boxed{\frac{1}{s^2}}$$

3. $f(t) = e^{at}$, where a is a constant.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}e^{-st}dt$$

$$= \int_0^\infty e^{(a-s)t}dt = \boxed{\frac{1}{s-a}}$$

4. $f(t) = \begin{cases} t^2, & 0 < t < 1, \\ 1, & 1 \le t \le 2, \\ 1-t, & 2 < t. \end{cases}$



$$\mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^1 t^2 e^{-st}\,dt + \int_1^2 e^{-st}\,dt + \int_2^\infty (1-t)e^{-st}\,dt$$

$$\left| \begin{array}{ll} u = t^2 & dv = e^{-st}dt \\ du = 2t\,dt & v = -\frac{1}{s}e^{-st} \end{array} \right| \qquad \left| \begin{array}{ll} u = 1-t & dv = e^{-st}dt \\ du = -dt & v = -\frac{1}{s}e^{-st} \end{array} \right|$$

$$= -\frac{t^2}{s}e^{-st}\Big]_{t=0}^{t=1} + \frac{2}{s}\int_0^1 te^{-st}\,dt - \frac{1}{s}e^{-st}\Big]_{t=1}^{t=2} + \lim_{N\to\infty}\left(\frac{t-1}{s}e^{-st}\Big]_{t=2}^{t=N} - \int_2^N \frac{1}{s}e^{-st}\,dt\right)$$

$$\left| \begin{array}{ll} u = t & dv = e^{-st}dt \\ du = dt & V = -\frac{1}{s}e^{-st} \end{array} \right|$$

$$= -\frac{1}{s}e^{-s} + \frac{2}{s}\left(-\frac{t}{s}e^{-st}\Big]_{t=0}^{t=1} + \frac{1}{s}\int_0^1 e^{-st}\,dt\right) - \frac{1}{s}\left(e^{-2s} - e^{-s}\right)$$

$$-\frac{1}{s}e^{-2s} + \frac{1}{s^2}\lim_{N\to\infty} e^{-st}\Big]_{t=2}^{t=N}$$

$$= \cancel{-\frac{1}{s}e^{-s}} + \frac{2}{s}\left(-\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-st}\Big]_{t=0}^{t=1}\right) - \frac{1}{s}\left(e^{-2s} - \cancel{e^{-s}}\right) - \frac{1}{s}e^{-2s}$$

$$-\frac{1}{s^2}e^{-2s}$$

$$= \boxed{-\frac{2}{s^2}e^{-s} - \frac{2}{s^3}e^{-s} + \frac{2}{s^3} - \frac{2}{s}e^{-2s} - \frac{1}{s^2}e^{-2s}}$$

Brief table of Laplace transform

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$

The important property of the Laplace transform is its **linearity**. That is, the Laplace transform $\mathcal{L}$ is a linear operator.

Theorem 1. **(linearity of the transform)** Let f_1 and f_2 be functions whose Laplace transform exist for $s > \alpha$ and c_1 and c_2 be constants. Then, for $s > \alpha$,

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\}.$$

Example 2. Determine $\mathcal{L}\{10 + 5e^{2t} + 3\cos 2t\}$.

$$= \mathcal{L}\{10\} + \mathcal{L}\{5e^{2t}\} + \mathcal{L}\{3\cos 2t\}$$

$$= 10\,\mathcal{L}\{1\} + 5\,\mathcal{L}\{e^{2t}\} + 3\,\mathcal{L}\{\cos 2t\}$$

$$= \boxed{\frac{10}{s} + \frac{5}{s-2} + \frac{3s}{s^2+4}}$$

Existence of the transform.

There are functions for which the improper integral in Definition 1 fails to converge for any value of s. For example, no Laplace transform exists for the function e^{t^2}. Fortunately, the set of the functions for which the Laplace transform is defined includes many of the functions.

Definition 2. A function f is said to be **piecewise continuous on a finite interval** $[a, b]$ if f is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(x)$ is said to be **piecewise continuous on** $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

Definition 3. A function $f(t)$ is said to be of **exponential order** α if there exist positive constants T and M s.t.

$$|f(t)| \le Me^{\alpha t}, \text{ for all } t \ge T.$$

Theorem 2. If $f(t)$ is piecewise continuous on $t \to \infty$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

Properties of Laplace transform

1. $\mathcal{L}\{f + g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
2. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$ for any constant c
3. $\mathcal{L}\{e^{at} f\}(s) = F(s - a)$
4. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
5. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
6. $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$
7. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n}{ds^n}(\mathcal{L}\{f(t)\})(s)$

Example. Find the Laplace transform of

$\boxed{\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}}$

(a) $f(t) = t^2 e^{-2t}$

$$\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3} = F(s)$$

$$\boxed{\mathcal{L}\{t^2 e^{-2t}\} = \frac{2}{(s+2)^3}} = F(s+2)$$

(b) $f(t) = t \sin 5t$

$\mathcal{L}\{\sin 5t\} = \dfrac{5}{s^2+25}$

$$\mathcal{L}\{t \sin 5t\} = (-1)\left[\mathcal{L}\{\sin 5t\}\right]'$$

$$= (-1)\left(\frac{5}{s^2+25}\right)' = (-1)\,5\left[(s^2+25)^{-1}\right]'$$

$$= (-1)(5)(-1)(s^2+25)^{-2}(2s)$$

$$= \boxed{\frac{10s}{(s^2+25)^2}}$$

Inverse Laplace Transform.

Definition 3. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0,\infty)$ and satisfies

$$\mathcal{L}\{f\}(s) = F(s),$$

then we say that $f(t)$ is the **inverse** Laplace transform of $F(s)$ and employ the notation $f(t) = \mathcal{L}^{-1}\{F\}(t)$.

Example 3. Determine the inverse Laplace transform of the given function.

1. $F(s) = \dfrac{2}{s^3}$.

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$3=2+1$

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n \xrightarrow{n=2} \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \boxed{t^2}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{3\cdot 2!}{2!\,s^3}\right\} = \frac{3}{2}t^2}$$

2. $F(s) = \dfrac{2}{s^2+4}$.

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$$

$$\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin bt$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \boxed{\sin 2t}$$

3. $F(s) = \dfrac{s+1}{s^2+2s+10}$. $= \dfrac{s+1}{(s+1)^2+9}$

$s^2+2s+10$ – quadratic irreducible factor

complete the square

$s^2+2s+10 = (s+1)^2+9$

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} = \boxed{e^{-t}\cos 3t}$$

4. $F(s) = \dfrac{s}{s^2+s-2}$,

$$= \frac{s}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

$$\frac{s}{\cancel{(s+2)(s-1)}} = \frac{A(s-1)+B(s+2)}{\cancel{(s+2)(s-1)}}$$

$$s = A(s-1)+B(s+2)$$

$s=1$: $\quad 1 = 3B$

$s=-2$: $\quad -2 = -3A$

$$\boxed{A = 2/3 \quad B = 1/3}$$

$$\frac{s}{s^2+s-2} = \frac{2}{3}\cdot\frac{1}{s+2} + \frac{1}{3}\,\frac{1}{s-1}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s-2}\right\} = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \boxed{\frac{2}{3}e^{-2t} + \frac{1}{3}e^{t}}$$

5. $F(s) = \dfrac{3s^2+5s+3}{s^4-s^2}$

$$s^4-s^2 = s^2(s^2-1) = s^2(s-1)(s+1)$$

$$\frac{3s^2+5s+3}{s^4-s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}$$

$$= \frac{As(s-1)(s+1) + B(s-1)(s+1) + C\,s^2(s+1) + Ds^2(s-1)}{s^2(s-1)(s+1)}$$

$$3s^2+5s+3 = As(s-1)(s+1) + B(s-1)(s+1) + Cs^2(s+1) + Ds^2(s-1)$$

$s=0$: $\quad 3 = -B \Rightarrow \boxed{B=-3}$

$s=1$: $\quad 11 = 2C \Rightarrow \boxed{C = 11/2}$

$s=-1$: $\quad 1 = -2D \Rightarrow \boxed{D = -1/2}$

$s=2$: $\quad 25 = A(2)(3) + B(3) + C(4)(3) + 4D$

$$25 = 6A - 9 + 66 - 2$$

$$6A = 25 + 11 - 66$$

$$6A = -30 \Rightarrow \boxed{A=-5}$$

$$\mathcal{L}^{-1}\left\{\frac{3s^2+5s+3}{s^4-s^2}\right\} = -5\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{11}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$-\frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \boxed{-5 - 3t + \frac{11}{2}e^{t} - \frac{1}{2}e^{-t}}$$