Section 6.6 **The convolution integral.**

Theorem. If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \quad s > a,$$

where

$$\boxed{\mathcal{L}^{-1}\{H(s)\} = h(t) = \int_0^1 f(t-\tau)g(\tau)d\tau = \int_0^1 f(\tau)g(t-\tau)d\tau.}$$

The function h is known as a convolution of f and g ($h(t) = (f * g)(t)$); the integrals in the formula for $h(t)$ are known as convolution integrals.

Properties of the convolution.

1. $f * g = g * f$
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0 * f = 0$

Example 1. Find the Laplace transform of the function

$$f(t) = \int_0^1 (t-\tau)^2 \cos 2\tau \, d\tau = (g * h)(t)$$

$$= \int_0^1 g(t-\tau)\, h(\tau)\, d\tau$$

where $g(t-\tau) = (t-\tau)^2 \Rightarrow g(t) = t^2$

$h(\tau) = \cos 2\tau \Rightarrow h(t) = \cos 2t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}\,\mathcal{L}\{h(t)\}$$

$$\mathcal{L}\{f(t)\} = \underbrace{\mathcal{L}\{t^2\}}_{\frac{2}{s^3}}\,\underbrace{\mathcal{L}\{\cos 2t\}}_{\frac{s}{s^2+4}}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2}{s^3} \cdot \frac{s}{s^2+4}}$$

Example 2. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4(s^2+1)}\right\}$ $= \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}}_{f(t)} \cdot \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}}_{g(t)}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4(s^2+1)}\right\} = (f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$\mathcal{L}^{-1}\left\{\frac{(1)6}{6s^4}\right\} = \frac{t^3}{6} = f(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = g(t)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^4(s^2+1)}\right\} = \int_0^t \frac{(t-\tau)^3}{6}\sin\tau\,d\tau = \int_0^t \frac{\tau^3}{6}\sin(t-\tau)\,d\tau}$$

Example 3. Express the solution of the initial value problem

$$y'' + \omega^2 y = g(t), \qquad y(0) = 0, y'(0) = 1$$

in terms of a convolution integral.

$$\mathcal{L}\{g(t)\} = G(s)$$
$$\mathcal{L}\{y(t)\} = Y(s)$$
$$\mathcal{L}\{y''\} = s^2Y(s) - 1$$

$$s^2Y(s) - 1 + \omega^2 Y(s) = G(s)$$
$$Y(s)(s^2+\omega^2) = G(s) + 1$$
$$Y(s) = \frac{G(s)}{s^2+\omega^2} + \frac{1}{s^2+\omega^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+\omega^2} + \frac{1}{s^2+\omega^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \sin(\omega t)$$

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+\omega^2}\right\} = \int_0^t g(t-\tau)\sin(\omega\tau)\,d\tau$$

$$\boxed{y(t) = \int_0^t g(t-\tau)\sin(\omega\tau)\,d\tau + \sin(\omega t)}$$