Section 7.2 **Review of Matrices.**

1. An $m \times n$ (this is often called the **size** or **dimension** of the matrix) matrix is a table with m rows and n columns and the entry in the i-th row and j-th column is denoted by a_{ij}:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

2. A matrix is usually denoted by a capital letter and its elements by small letters : $a_{ij} =$ entry in the ith row and jth column of A.

3. Two matrices are said to be **equal** if they are the same size and each corresponding entry is equal.

4. Special Matrices:

- A **square** matrix is any matrix whose size (or dimension) is $n \times n$ (i.e. it has the same number of rows as columns.) In a square matrix the diagonal that starts in the upper left and ends in the lower right is often called the **main diagonal.**
- The **zero** matrix is a matrix all of whose entries are zeroes.
- The **identity** matrix is a square $n \times n$ matrix, denoted I_n, whose main diagonal consists of all 1's and all the other elements are zero:

$$I_n = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

- The **diagonal** matrix is a square $n \times n$ matrix of the following form

$$\begin{pmatrix} \lambda_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \lambda_n \end{pmatrix}$$

- **Column matrix** (=column vector) and the **row matrix** (=row vector) are those matrices that consist of a single column or a single row respectively:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}$$

  Note that an n-dimensional column vector is an $n \times 1$ matrix, and an n-dimensional row vector is an $1 \times n$ matrix.

- **Transpose of a Matrix**: If A is an $m \times n$ matrix with entries a_{ij} , then A^T is the $n \times m$ matrix with entries a_{ji}.
  A^T is obtained by interchanging rows and columns of A.

5. **Matrix Arithmetic**

- The **sum** or difference of two matrices of the same size is a new matrix of identical size whose entries are the sum or difference of the corresponding entries from the original two matrices. Note that we cant add or subtract entries with different sizes.
- The **scalar multiplication** by a constant gives a new matrix whose entries have all been multiplied by that constant.
- If A, B, and C are matrices of the *same* size, then
  (a) $A + B = B + A$ (Commutative Law)
  (b) $(A + B) + C = A + (B + C)$ (Associative Law)
- **Matrix multiplication**: If Y is a row matrix of size $1 \times n$ and X is a column matrix of size $n \times 1$ (see above), then the **matrix product** of Y and X is defined by

$$YX = \boxed{\begin{pmatrix} y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_nx_n}$$

- If A is an $m \times p$ matrix and matrix B is $p \times n$, then the product AB is an $m \times n$ matrix, and its element in the ith row and jth column is the product of the ith row of A and the jth column of B.
- RULE for multiplying matrices:
  The column of the 1st matrix must be the same size as the row of the 2nd matrix.

$$A = \overset{m\times n}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}} \; ; \quad B = \overset{n\times p}{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}}$$

$$C = AB = \overset{m\times p}{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1n}b_{n2}$$

$$\boxed{\begin{gathered} c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} \\ i = 1, \dots, m \\ j = 1, \dots, p \end{gathered}}$$

6. **Example 1.**

(a) Given

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -2 \\ 3 & 4 \\ 1 & 2 \end{pmatrix}$$

Compute $A - 2B$.

(b) Let $A = \begin{pmatrix} 1 & 2 & -3 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \end{pmatrix}$. Find BA^T.

$$A^T = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 0 \\ -1 \end{pmatrix}$$

$$BA^T = \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \\ 0 \\ -1 \end{pmatrix} = 1(1) + 2(2) + 0(-3) + 1(0) + (-1)(-1)$$

$$= \boxed{6}$$

(c) Given

$$A = \overset{3\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -1 & -2 \end{pmatrix}}, \quad B = \overset{2\times 2}{\begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}}$$

Compute AB and BA when it is possible.

2 columns $\neq$ 3 rows

$$BA = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -1 & -2 \end{pmatrix}$$ not possible

A (1st row)(2nd column) B

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1(-1)+2(3) & 1(-2)+2(4) \\ 3(-1)+4(3) & 3(-2)+4(4) \\ -1(-1)+(-2)(3) & (-1)(-2)+(-2)(4) \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 5 & 6 \\ 9 & 10 \\ -5 & -6 \end{pmatrix}}$$

(d) Compute $\begin{pmatrix} 1 & 2 & 5 \\ 3 & 2 & -3 \\ 4 & 3 & 9 \end{pmatrix} \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{I_3 \text{ - identity matrix}}$

$$= \begin{pmatrix} 1(1)+2(0)+5(0) & 1(0)+2(1)+5(0) & 1(0)+2(0)+5(1) \\ 3(1)+2(0)+(-3)(0) & 3(0)+2(1)+(-3)(0) & 3(0)+2(0)+(-3)(1) \\ 4(1)+3(0)+9(0) & 4(0)+3(1)+9(0) & 4(0)+3(0)+9(1) \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 & 2 & 5 \\ 3 & 2 & -3 \\ 4 & 3 & 9 \end{pmatrix}}$$

A is $n \times n$ matrix

$$\boxed{A I_n = I_n A = A}$$

(e) Compute AX if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ -1 & -2 & -3 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$AX = \overset{3\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ -1 & -2 & -3 \end{pmatrix}} \overset{3\times 1}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \overset{3\times 1}{\underset{\text{column vector}}{\begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ 3x_1 + 4x_2 + 5x_3 \\ -x - 2x_2 - 3x_3 \end{pmatrix}}}$$

7. FACTS and LAWS FOR MATRIX MULTIPLICATION: If the size requirements are met for matrices A, B and C, then

- $AB \neq BA$ (NOT always Commutative) (Since the multiplication of matrices is NOT commutative, you MUST multiply left to right.)
- $A(B+C) = AB + AC$ (always Distributive)
- $(AB)C = A(BC)$ (always Associative)
- $AB = 0$ does not imply that $A = 0$ or $B = 0$.
- $AB = AC$ does not imply that $B = C$.
- $\boxed{I_n A = A I_n = A \text{ for any square matrix } A \text{ of size } n.}$

8. A system of linear equations can be written as a matrix equation $AX = B$.

9. **Example 2.** Express the following system of linear equations in matrix form:

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 7x_3 & = & 6 \\ -x_1 & - & 3x_2 & + & 11x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & 1 \end{array}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}$$

coefficient matrix. $A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 2 & 4 & -7 \\ -1 & -3 & 11 \\ 0 & -1 & 1 \end{pmatrix} \end{array}$

matrix form: $\boxed{\begin{pmatrix} 2 & 4 & -7 \\ -1 & -3 & 11 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}}$

10. **Trace** of a square matrix is equal to the sum of its diagonal elements: If A is a **square matrix** of size n then

$$\text{trace}(A) = a_{11} + a_{22} + \ldots + a_{nn}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

main diagonal

Determinant

11. Determinant of a matrix is a function that takes a square matrix and converts it into a number.

12. Determinant of 2×2 and 3×3 matrices.

- A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- A **determinant of order 3** is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$
$$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1$$

$\det A = 0$

- If the determinant of a matrix is zero we call that matrix singular and if the determinant of a matrix isnt zero we call the matrix nonsingular.

$\det A \neq 0$

determinant of a matrix isn't zero we call the matrix **nonsingular.**

13. **Matrix Inverse.** Let A be a square matrix of size n. A square matrix, A^{-1}, of size n, such that $\boxed{AA^{-1} = I_n}$ (or , equivalently, $\boxed{A^{-1}A = I_n}$) is called an inverse matrix.

14. A^{-1} in the case $n = 2$: If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then

$$\boxed{A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}$$

15. *FACT*: A^{-1} exists if and only if $\det A \neq 0$.

Equivalently:

- If A is nonsingular then A^{-1} will exist.
- If A is singular then A^{-1} will NOT exist.

16. **Solving Systems of Equations with Inverses.**

Let $AX = B$ be a linear system of n equations in n unknowns and A^{-1} exists, then $\boxed{X = A^{-1}B}$ is the *unique* solution of the system.

Example 3. solve the system:

$$\begin{cases} 3x_1 + 7x_2 = 5 \\ 7x_1 - 15x_2 = 6 \end{cases}$$

matrix form: $\underbrace{\begin{pmatrix} 3 & 7 \\ 7 & -15 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$$\vec{x} = A^{-1}\vec{b}$$

$A = \begin{pmatrix} 3 & 7 \\ 7 & -15 \end{pmatrix}$, $\det A = 3(-15) - 49 = -94$

$$A^{-1} = -\frac{1}{94}\begin{pmatrix} -15 & -7 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} \frac{15}{94} & \frac{7}{94} \\ \frac{7}{94} & -\frac{3}{94} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{15}{94} & \frac{7}{94} \\ \frac{7}{94} & -\frac{3}{94} \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5(15) + 7(6)}{94} \\ \frac{7(5) - 3(6)}{94} \end{pmatrix} = \begin{pmatrix} \frac{117}{94} \\ \frac{17}{94} \end{pmatrix}$$

$$\boxed{x_1 = \frac{117}{94}, \; x_2 = \frac{17}{94}}$$