Section 7.5 **Homogeneous linear systems with constant coefficients**
Section 7.6 **Complex eigenvalues**
Section 7.8 **Repeated eigenvalues**

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

here $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$.

- **Real Distinct Eigenvalues.** If A has n **distinct real** eigenvalues $\lambda_1, \dots, \lambda_n$ and $v_1, \dots, v_n$ are the corresponding eigenvectors, then

$$\{e^{\lambda_1 t} v_1, \dots, e^{\lambda_n t} v_n\}$$

is the fundamental solution set and the general solution is

$$X(t) = C_1 e^{\lambda_1 t} v_1 + \dots + C_n e^{\lambda_n t} v_n.$$

Example 1. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}, \quad A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}$$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 4 = 0$$

$$-6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$\lambda_1 = 2, \ \lambda_2 = -1$ eigenvalues.

Corresponding eigenvectors:

1) $\lambda_1 = 2$ $\quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{v}$ is a solution of

$$(A - 2I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad \begin{cases} v_1 - 2v_2 = 0 \\ 2v_1 - 4v_2 = 0 \end{cases}$$

$$v_1 = 2v_2$$

$$\vec{v} = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} \underset{=}{\overset{v_2 = 1}{=}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

2) $\lambda_2 = -1$ $\quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a solution of

$$(A + I)\vec{w} = \vec{0}$$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{cases} 4w_1 - 2w_2 = 0 \\ 2w_1 - w_2 = 0 \end{cases}$$

$$w_2 = 2w_1$$

$$\vec{w} = \begin{pmatrix} w_1 \\ 2w_1 \end{pmatrix} \underset{=}{\overset{w_1 = 1}{=}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

General solution: $\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}}$

- **Complex eigenvalues.** Any complex eigenvalue must occur in conjugate pairs. If $\lambda = \alpha + i\beta$ is an eigenvalue of the matrix $\mathbf{A}$, then so is $\bar\lambda = \alpha - i\beta$.

  If $\vec v = \vec u + i\vec w$ is an eigenvector corresponding to $\lambda = \alpha + i\beta$, then $\bar{\vec v} = \vec u - i\vec w$ is an eigenvector corresponding to $\bar\lambda = \alpha - i\beta$. Then the corresponding solution of the system is

$$\vec x(t) = (\vec u + i\vec w)e^{(\alpha+i\beta)t} = (\vec u + i\vec w)(\cos\beta t + i\sin\beta t)e^{\alpha t}$$
$$= (\vec u\cos\beta t - \vec w\sin\beta t)e^{\alpha t} + i\,(\vec u\sin\beta t + \vec w\cos\beta t)e^{\alpha t}$$

Then the vectors

$$\boxed{\begin{aligned}\vec x_1(t) &= Re(\vec x(t)) = (\vec u\cos\beta t - \vec w\sin\beta t))e^{\alpha t}\\ \vec x_2(t) &= Im(\vec x(t)) = (\vec u\sin\beta t + \vec w\cos\beta t))e^{\alpha t}\end{aligned}}$$

are real valued solutions of the system.

Example 2. Find the general solution of the system

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\mathbf{x}$ $\quad A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix},\ A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix}$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & (-1-\lambda) \end{vmatrix} = 0$$
$$(3-\lambda)(-1-\lambda) + 8 = 0$$
$$-3 - 3\lambda + \lambda + \lambda^2 + 8 = 0$$
$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda_1 = \frac{2 + \sqrt{4-20}}{2} = \frac{2+\sqrt{-16}}{2} = \boxed{1 + 2i = \lambda_1}$$
$$\boxed{\lambda_2 = 1 - 2i}$$

Corresponding eigenvectors. $(\lambda_1 = 1+2i)\ \vec v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of $(A - \overbrace{\lambda_1}^{1+2i} I)\vec v = \vec 0$

$$\begin{pmatrix} 3-(1+2i) & -2 \\ 4 & -1-1-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Component form: $\begin{cases} (2-2i)v_1 - 2v_2 = 0 \\ 4v_1 - (2+2i)v_2 = 0 \end{cases}$ $\quad \boxed{v_2 = (1-i)\,v_1}$

$$\vec v = \begin{pmatrix} v_1 \\ (1-i)v_1 \end{pmatrix} \underset{=}{\overset{v_1=1}{=}} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\boxed{e^{(1+2i)t} = e^t(\cos 2t + i\sin 2t)}$$

$$\vec v e^{\lambda_1 t} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]e^{(1+2i)t}$$
$$= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]e^t(\cos 2t + i\sin 2t)$$
$$= e^t\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} i\sin 2t + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos 2t + \overbrace{i^2}^{-1}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t\right]$$
$$= e^t\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t + i\left\{\begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin 2t\right\}\right]$$
$$= e^t\left[\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}\right]$$

$$\vec x_1 = Re(\vec v e^{\lambda_1 t}) = e^t\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}$$
$$\vec x_2 = Im(\vec v e^{\lambda_1 t}) = e^t\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$

General solution: $\vec x(t) = c_1\vec x_1 + c_2\vec x_2 = \boxed{c_1 e^t\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}}$

$$\vec{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}$$

eigenvalues: $\lambda_1 = -1+2i, \quad \lambda_2 = -1-2i$

corresponding eigenvector $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$e^{(-1+2i)t} = e^{-t}(\cos 2t + i \sin 2t)$$

$$\vec{v} e^{\lambda_1 t} = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right] e^{(-1+2i)t}$$

$$= \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right] e^{-t} (\cos 2t + i \sin 2t)$$

$$= e^{-t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t + i \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t + \overbrace{i^2}^{-1} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right]$$

$$= e^{-t} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t + i \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right] \right\}$$

$$= e^{-t} \left(\begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix} \right)$$

$$\vec{x}_1(t) = \mathrm{Re}\left(\vec{v} e^{\lambda_1 t} \right) = e^{-t} \begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\vec{x}_2(t) = \mathrm{Im}\left(\vec{v} e^{\lambda_1 t} \right) = e^{-t} \begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix}$$

General solution: $\boxed{\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix}}$

- **Repeated eigenvalues.** Let λ_i be an eigenvalue of the matrix $\mathbf{A}$ of multiplicity $1 < k \leq n$. Then there are k linearly independent eigenvectors $\vec{v}_1(t)$,...,$\vec{v}_1(t)$ corresponding to λ, if $k < n$. If $k = n$, then there is only one vector $\vec{v}_1(t)$ corresponding to λ. The remaining $n-1$ vectors corresponding to λ are solutions to the system

$$(\mathbf{A} - \lambda \mathbf{I})\vec{v}_{i+1}(t) = \vec{v}_i(t), \quad i = 1, 2, ..., n-1$$

Vectors $\vec{v}_2(t)$, ... $\vec{v}_{n-1}(t)$ are called **generalized eigenvectors** corresponding to λ.

Then the corresponding solutions of the system are $v_1(t)e^{\lambda t}$, $tv_2(t)e^{\lambda t}$,..., $t^k v_k(t)e^{\lambda t}$.

Example 3. Find the general solution of the system.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$ $\qquad A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix}$

Eigenvalues: $\quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 4 = 0$

$$-3 - 3\lambda + \lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$\lambda = 1$ - repeated eigenval

Corresponding eigenvectors. $\quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of

$$(A - \overset{1}{\lambda} I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad \begin{cases} 2v_1 - 4v_2 = 0 \\ v_1 - 2v_2 = 0 \end{cases}, \quad v_1 = 2v_2$$

$$\vec{v} = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \overset{v_2 = 1}{=} \boxed{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

2nd eigenvector $\quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is a solution of

$$(A - I)\vec{y} = \vec{v}$$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; \quad \begin{cases} 2y_1 - 4y_2 = 2 \\ y_1 - 2y_2 = 1 \end{cases}$$

$$y_1 = 1 + 2y_2$$

$$\vec{y} = \begin{pmatrix} 1 + 2y_2 \\ y_2 \end{pmatrix} \overset{y_2 = 0}{=} \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

General solution: $\boxed{\vec{x}(t) = \left[c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^t}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}$

$A = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ -4 & 1-\lambda & 0 \\ 3 & 6 & 2-\lambda \end{pmatrix}$

Eigenvalues. $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -4 & 1-\lambda & 0 \\ 3 & 6 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) = 0$

$\lambda_1 = 1$ - repeated of multiplicity 2

$\lambda_2 = 2$ - nonrepeated

Corresponding eigenvectors.

1) $\lambda_2 = 2$ corresponding eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is a solution of $(A - \overbrace{\lambda_2}^{2} I)\vec{v} = \vec{0}$

$$\begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \begin{cases} -v_1 = 0 \\ -4v_1 - v_2 = 0 \\ 3v_1 + 6v_2 = 0 \end{cases}$$

$v_1 = v_2 = 0$, v_3 can be any number

$$\vec{v} = \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \underset{=}{\overset{v_3=1}{=}} \boxed{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = 2}$$

2) $\lambda_1 = 1$ - repeated, of multiplicity 2.

$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ is a solution of $(A - \overbrace{\lambda_1}^{1} I)\vec{w} = \vec{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \begin{cases} -4y_1 = 0 \\ 3y_1 + 6y_2 + y_3 = 0 \end{cases}$$

$y_1 = 0$, $y_3 = -6y_2$

$$\vec{y} = \begin{pmatrix} 0 \\ y_2 \\ -6y_2 \end{pmatrix} \underset{=}{\overset{y_2=1}{=}} \boxed{\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}}$$

2nd eigenvector corresponding to $\lambda_1 = 1$ $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ is a solution of the system $(A - I)\vec{w} = \vec{y}$

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}; \quad \begin{cases} -4w_1 = 1 \\ 3w_1 + 6w_2 + w_3 = -6 \end{cases}$$

$w_1 = -1/4$

$w_3 = -6 - 3w_1 - 6w_2$

$= -6 + \frac{3}{4} - 6w_2$

$w_3 = -\frac{21}{4} - 6w_2$

$$\vec{w} = \begin{pmatrix} -1/4 \\ w_2 \\ -\frac{21}{4} - 6w_2 \end{pmatrix} \underset{=}{\overset{w_2=0}{=}} \boxed{\begin{pmatrix} -1/4 \\ 0 \\ -\frac{21}{4} \end{pmatrix}}$$

General solution: $\boxed{\vec{x}(t) = \left[c_1 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} + c_2 t \begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix}\right] e^{t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

$A = \begin{pmatrix} 1 & -4 \\ 1 & -1 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 1-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix}$

Eigenvalues. $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) + 4 = 0$

$$-1 + \lambda^2 + 4 = 0$$

$$\lambda^2 + 3 = 0$$

$$\boxed{\lambda_1 = \sqrt{3}\,i \quad \lambda_2 = -\sqrt{3}\,i}$$

Corresponding eigenvector $\lambda_1 = \sqrt{3}\,i$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of

$$\begin{pmatrix} 1-\sqrt{3}\,i & -4 \\ 1 & -1-\sqrt{3}\,i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (1-\sqrt{3}\,i)\,v_1 - 4 v_2 = 0 \\ v_1 - (1+\sqrt{3}\,i)\,v_2 = 0 \end{cases} \qquad \boxed{v_1 = (1+\sqrt{3}\,i)\,v_2}$$

$$\vec{v} = \begin{pmatrix} v_2(1+\sqrt{3}\,i) \\ v_2 \end{pmatrix} \overset{v_2 = 1}{=} \begin{pmatrix} 1+\sqrt{3}\,i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \sqrt{3}\,i \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}}$$

$$\vec{v}\, e^{\lambda_1 t} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \right] e^{\sqrt{3}\,i t}$$

$$e^{\sqrt{3}\,i t} = e^{0 \cdot t}(\cos\sqrt{3}t + i \sin\sqrt{3}t) = \cos\sqrt{3}t + i \sin\sqrt{3}t$$

$$= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \right] (\cos\sqrt{3}t + i \sin\sqrt{3}t)$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\sqrt{3}t + i \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \cos\sqrt{3}t + i \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin\sqrt{3}t + \overbrace{i^2}^{-1} \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \sin\sqrt{3}t$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\sqrt{3}t - \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \sin\sqrt{3}t + i \left[\begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix} \cos\sqrt{3}t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin\sqrt{3}t \right]$$

$$= \begin{pmatrix} \cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \\ \cos\sqrt{3}t \end{pmatrix} + i \begin{pmatrix} \sqrt{3}\cos\sqrt{3}t + \sin\sqrt{3}t \\ \sin\sqrt{3}t \end{pmatrix}$$

General solution: $\boxed{\vec{x}(t) = c_1 \begin{pmatrix} \cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \\ \cos\sqrt{3}t \end{pmatrix} + c_2 \begin{pmatrix} \sqrt{3}\cos\sqrt{3}t + \sin\sqrt{3}t \\ \sin\sqrt{3}t \end{pmatrix}}$