

## Chapter 1. Introduction.

- Equation that contains some derivatives of an unknown function is called a **differential equation**.
- If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter is called an **independent variable**.
- A differential equation involving only ordinary derivatives with respect to a single variable is called an **ordinary differential equations** or ODE. A differential equation involving partial derivatives with respect to more than one variable is a **partial differential equations** or PDE.

**Example 1.** For the following differential equations indicate independent and dependent variables.

1.  $y^V + y^2 y''' + y' \sin x = 0$

2.  $t^2 y'' + ty = 0$

3.  $x''' \sqrt{x} - 3t^2 x' + xe^t = 0$

- A differential equation that describes some physical process is often called a **mathematical model** of the process.

### Examples.

1. A falling object.
  - (a) Neglect the air resistance.

(b) Air resistance is proportional to the velocity of the object.

2. Field mice and owls. Consider a population of field mice who inhabit a certain rural area. In the absence of predators the rate of change of the mouse population is proportional to the current population.

If we assume that predator rate is a constant  $k$ , ( $k > 0$ ), then

### Constructing math models:

1. Identify the independent and dependent variables and assign letters to represent them.
2. Choose the units of measurement for each variable.
3. Articulate the basic principle that underlines or governs the problem you are investigating.
4. Express the principle or laws from Step 3 in terms of variables from Step 1.
5. Make sure that each term in the equation has the same physical units.
6. The result of Step 4 is a single differential equation which constitutes the math model.

**Example 2.** A pond initially contains  $10^6$  gallons of water and an unknown amount of an undesirable chemical. Water containing 0.01 g of this chemical per gallon flows into the pond at a rate of 300 gal/hour. The mixture flows out at the same rate. Assume that the chemical is uniformly distributed throughout the pond. Write a differential equation for the amount of chemical in the pond at any time.

- The **order** of a differential equation is the order of the highest-order derivatives present in equation.

**Example 3.** Determine the order of the following differential equations.

1.  $yy''' - 10y'' + y^V = 5y^{10}$
2.  $x'' + tx' - t^3x = \cos t$

- A **general form** for an  $n$ th-order equation with variable  $x$  and unknown function  $y = y(x)$  can be expressed as

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0, \quad (1)$$

where  $F$  is a function that depends on  $x$ ,  $y$ , and the derivatives of  $y$  up to the order  $n$ . We assume that the equation holds for all  $x$  in an open interval  $I$  ( $a < x < b$ , where  $a$  or  $b$  could be infinite).

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}}\right) \quad (2)$$

- A function  $\varphi(x)$  that when substituted by  $y$  in the equation satisfies the equation for all  $x$  in the interval  $I$  is called a **solution** to the equation on  $I$ .

**Example 4.** Determine whether a function is a solution to the given equation.

1.  $tg' = 2g, g(t) = 5t^2$

2.  $y^3y'' + 1 = 0, y(t) = \sqrt{2x - x^2}$

- By an **initial value problem** for an  $n$ th-order differential equation

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0$$

we mean: Find a solution to the differential equation on an interval  $I$  that satisfies at  $x_0$  the  $n$  initial conditions:

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = y_1, \dots, \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1},$$

where  $x_0 \in I$  and  $y_0, y_1, \dots, y_{n-1}$  are given constants.

- In case of a first-order equation  $F\left(x, y, \frac{dy}{dx}\right)$ , the initial conditions reduce to the single requirement

$$y(x_0) = y_0$$

- In case of a second-order equation, the initial conditions have the form

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = y_1.$$

- An ODE is **linear** if it has format

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + a_1(x) \frac{dy}{dx} + a_0(x)y = F(x),$$

where  $a_n(x), a_{n-1}(x), \dots, a_0(x)$  and  $F(x)$  depend only on variable  $x$ . If an ODE is not linear, then we call it **nonlinear**.

**Example 5.** For each of the differential equations indicate whether it is linear or nonlinear.

1.  $\ln(x) \frac{d^2 y}{dx^2} + 3e^x \frac{dy}{dx} - y \sin x = 0$

2.  $2y'' - 3y^2 = e^x$

3.  $\frac{d^3 y}{dx^3} + (x^2 - 1)y + \cos x = 0$

4.  $y'' - \sin(x + y)y' + (x^2 + 1)y = 0$

5.  $y'' - e^{xy} = \cos(2x + y)$

## Direction Fields.

One technique that is useful in graphing the solutions to a first-order differential equation is to sketch the direction field for the equation. To describe this method, we need to make a general observation.

Namely, a first-order equation

$$\frac{dy}{dx} = f(x, y)$$

specifies a slope at each point in the  $xy$ -plane where  $f$  is defined.

A plot of short line segments drawn at various points in the  $xy$ -plane showing the slope of the solution curve there is called a **direction field** for the differential equation. Because the direction field gives the "flow of solutions", it facilitates the drawing of any particular solution (such as the solution to an initial value problem).

### Example 6.

1. Sketch a direction field for  $y' = t + y$
2. Using the direction field, describe the behavior of the solution as  $t \rightarrow \pm\infty$ .

We can get directional field using MatLab.

```
>> [t,y]=meshgrid(-3:0.2:2, -1:0.2:5);
>> s=t+y;
>> quiver(t,y,ones(size(s)), s), axis tight
All array must have the same size, therefore we do
>> [t,y]=meshgrid(-3:0.2:2, -1:0.2:5);
>> s=t+y;
>> l=sqrt(1+s.^ 2);
>> quiver(t,y,1./l,s./l,0.5), axis tight
```

To graph solutions, we do

```
>> [t,y]=meshgrid(-3:0.2:2, -1:0.2:5);
>> s=t+y;
>> l=sqrt(1+s.^ 2);
>> quiver(t,y,1./l,s./l,0.5), axis tight
>> f=@(t,y)t+y;
>> hold on
>> for y0=-3:1:2
[ts,ys]=ode45(f, [-3,2], y0); plot(ts,ys)
end
>> hold off
```

### Example 7.

1. Sketch a direction field for  $y' = t^2 - y$
2. Using the direction field, describe the behavior of the solution as  $t \rightarrow \pm\infty$ .