1. Find the Laplace transform of the given function.

(a) $f(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$, $f(t) = \frac{t}{2} + \left(3 - \frac{t}{2}\right) u_6(t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{t}{2} + \left(3 - \frac{t}{2}\right) u_6(t)\right\}$$

$$= \mathcal{L}\left\{\frac{t}{2}\right\} + \mathcal{L}\left\{\frac{1}{2}(6-t) u_6(t)\right\}$$

$$= \frac{1}{2s^2} + \frac{1}{2}(-1)\,\mathcal{L}\{(t-6) u_6(t)\}$$

$$= \frac{1}{2s^2} - \frac{1}{2} e^{-6s} \mathcal{L}\{t\} = \boxed{\frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}}$$

(b) $f(t) = (t^2 - 2t + 2)u_1(t)$

$$f(t) = [(t-1)^2 + 1] u_1(t)$$

$$= (t-1)^2 u_1(t) + u_1(t)$$

$$\mathcal{L}\{(t-1)^2 u_1(t) + u_1(t)\}$$

$$= e^{-s} \mathcal{L}\{t^2\} + \frac{e^{-s}}{s}$$

$$= \boxed{e^{-s} \frac{2}{s^3} + \frac{e^{-s}}{s}}$$

(c) $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau = (g * h)(t)$

$g(t-\tau) = (t-\tau)^2 \Rightarrow g(t) = t^2$

$h(\tau) = \cos 2\tau \Rightarrow h(t) = \cos 2t$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \cdot \mathcal{L}\{h(t)\}$

$= \mathcal{L}\{t^2\} \cdot \mathcal{L}\{\cos 2t\}$

$= \boxed{\frac{2}{s^3} \cdot \frac{s}{s^2+4}}$

(d) $f(t) = e^t\delta(t-1)$

$$\mathcal{L}\{\delta(t-1)\} = e^{-s}$$

$$\mathcal{L}\{e^t\,\delta(t-1)\} = \boxed{e^{-(s-1)}}$$

2. Find the inverse Laplace transform of the given function.

(a) $F(s) = \dfrac{2s+6}{(s^2-4s+8)s^4} = G(s)\cdot H(s)$, where $G(s) = \dfrac{2s+6}{s^2-4s+8}$ and $H(s) = \dfrac{1}{s^4}$

$$\frac{2s+6}{s^2-4s+8} = \frac{2s+6}{(s-2)^2+4} = 2\,\frac{s+3}{(s-2)^2+4}$$

$$= 2\,\frac{s-2+5}{(s-2)^2+4} = 2\frac{s-2}{(s-2)^2+4} + 5\cdot\frac{2}{(s-2)^2+4}$$

$$\mathcal{L}^{-1}\{G(s)\} = 2\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+4}\right\}$$

$$= \boxed{2e^{2t}\cos 2t + 5e^{2t}\sin 2t}$$

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = \frac{t^3}{6}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{G(s)\} * \mathcal{L}^{-1}\{H(s)\}$$

$$= \boxed{\int_0^t \frac{(t-\tau)^3}{6}\left(2e^{2t}\cos 2t + 5e^{2t}\sin 2t\right)dt}$$

(b) $F(s) = \dfrac{e^{-2s}}{s^2+s-2}$

$$\frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$= \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

$$1 = A(s-1) + B(s+2)$$

$s=1$: $1 = 3B \Rightarrow B = 1/3$

$s=-2$: $1 = -3A \Rightarrow A = -1/3$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-2}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}\right\}$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+s-2}\right\} = \boxed{\left[-\frac{1}{3}e^{-2(t-2)} + \frac{1}{3}e^{(t-2)}\right]u_2(t)}$$

3. Solve the initial value problem using the Laplace transform:

(a) $y'' + 4y = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$, $y(0) = y'(0) = 0$

$g(t) = t + (1-t)u_1(t)$

$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - \mathcal{L}\{(t-1)u_1(t)\}$

$= \frac{1}{s^2} - e^{-s}\mathcal{L}\{t\}$

$= \frac{1}{s^2} - e^{-s}\frac{1}{s^2}$

$\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$

$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$

$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\}$

$(s^2+4)Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$

$Y(s) = \frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)}$

Partial fractions:

$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$

$= \frac{As(s^2+4) + B(s^2+4) + (Cs+D)s^2}{s^2(s^2+4)}$

$1 = s^3(A+C) + s^2(B+D) + s(4A) + 4B$

s^3: $0 = A + C$

s^2: $0 = B + D$

s: $0 = 4A$

1: $1 = 4B$

$\Longrightarrow$ $A = C = 0$, $B = 1/4$, $D = -1/4$

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}t - \frac{1}{8}\sin 2t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2+4)}\right\} = u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1)\right]$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)}\right\}$$

$$= \boxed{\frac{1}{4}t - \frac{1}{8}\sin 2t + u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1)\right]}$$

(b) $y'' + 2y' + 3y = \delta(t - 3\pi)$, $y(0) = y'(0) = 0$

$\mathcal{L}\{y'' + 2y' + 3y\} = \mathcal{L}\{\delta(t-3\pi)\}$

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = sY(s) - y(0)$

$= sY(s)$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$

$= s^2 Y(s)$

$\mathcal{L}\{\delta(t-3\pi)\} = e^{-3\pi s}$

$(s^2+2s+3)\,Y(s) = e^{-3\pi s}$

$Y(s) = \dfrac{1}{s^2+2s+3}\, e^{-3\pi s}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\dfrac{e^{-3\pi s}}{s^2+2s+3}\right\}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+3}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2+2}\right\}$

$= \dfrac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\dfrac{\sqrt{2}}{(s+1)^2+2}\right\}$

$= \dfrac{1}{\sqrt{2}} e^{-t} \sin\sqrt{2}\,t$

$\mathcal{L}^{-1}\left\{\dfrac{e^{-3\pi s}}{s^2+2s+3}\right\}$

$= \boxed{u_{3\pi}(t)\,\dfrac{1}{\sqrt{2}}\, e^{-(t-3\pi)} \sin\sqrt{2}(t-3\pi) = y(t)}$

(c) $y'' + 4y' + 4y = g(t)$, $y(0) = 2$, $y'(0) = -3$

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{g(t)\}$$
$$\mathcal{L}\{g(t)\} = G(s)$$
$$\mathcal{L}\{y\} = Y(s)$$
$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2$$
$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$
$$= s^2Y(s) - 2s + 3$$
$$s^2Y(s) - 2s + 3 + 4sY(s) - 8 + 4Y(s) = G(s)$$
$$Y(s)(s^2+4s+4) = G(s) + 2s + 5$$
$$Y(s) = \frac{G(s)}{s^2+4s+4} + \frac{2s+5}{s^2+4s+4}$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+4s+4} + \frac{2s+5}{s^2+4s+4}\right\}$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$
$$= e^{-2t}t$$
$$\frac{2s+5}{s^2+4s+4} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$
$$= \frac{A(s+2) + B}{(s+2)^2}$$

$$2s+5 = A(s+2) + B$$
$s=-2$: $1 = B$
$s=0$: $5 = 2A + B$, $2A = 5 - B = 4$
$A = 2$

$$\frac{2s+5}{s^2+4s+4} = \frac{2}{s+2} + \frac{1}{(s+2)^2}$$
$$\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+4s+4}\right\} = 2e^{-2t} + e^{-2t}t$$
$$\boxed{y(t) = \int_0^t g(t-\tau)e^{-2\tau}\tau\,d\tau + 2e^{-2t} + e^{-2t}t}$$

4. Find A^{-1} if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$

$$\det A = (1+i)(2-i) - (-1+2i)(3+2i) = 2 - i + 2i - \overbrace{i^2}^{-1} + 3 + 2i - 6i - 4\overbrace{i^2}^{-1}$$

$$= 5 + 5 - 3i = 10 - 3i$$

$$\frac{1}{\det A} = \frac{1}{10-3i} = \frac{10+3i}{(10-3i)(10+3i)} = \frac{10+3i}{100-9i^2} = \frac{10+3i}{109}$$

$$\boxed{A^{-1} = \frac{10+3i}{109}\begin{pmatrix} 2-i & 1-2i \\ -3-2i & 1+i \end{pmatrix}}$$

5. Find BA if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$, $B = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$

$$BA = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}\begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix} = \begin{pmatrix} i(1+i)+3(3+2i) & i(-1+2i)+3(2-i) \\ 2(1+i)-2i(3+2i) & 2(-1+2i)-2i(2-i) \end{pmatrix}$$

$$= \begin{pmatrix} i+i^2+9+6i & -i+2i^2+6-3i \\ 2+2i-6i-4i^2 & -2+4i-4i+4i^2 \end{pmatrix} = \boxed{\begin{pmatrix} 8+7i & 4-4i \\ 6-4i & -6 \end{pmatrix}}$$

6. Find the general solution of the system

(a) $x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x$ $\qquad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}, \; A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix}$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = 0$$

$$-2 - \lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda + 3)(\lambda - 2) = 0$$

$\lambda_1 = -3, \; \lambda_2 = 2$ – eigenvalues

Corresponding eigenvectors:

$\lambda_1 = -3$. Corresponding eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of the system $(A + 3I)\vec{v} = \vec{0}$

solution

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$4v_1 + v_2 = 0 \Rightarrow v_2 = -4v_1$

$\vec{v} = \begin{pmatrix} v_1 \\ -4v_1 \end{pmatrix} \overset{v_1 = 1}{=} \boxed{\begin{pmatrix} 1 \\ -4 \end{pmatrix} \text{ corresponds to } \lambda_1 = -3}$

$\lambda_2 = 2$. $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$(A - 2I)\vec{w} = \vec{0}$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-w_1 + w_2 = 0 \Rightarrow w_1 = w_2$

$\vec{w} = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix} \overset{w_1 = 1}{=} \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = 2}$

General solution:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}}$$

(c) $x' = \cancel{\begin{pmatrix} 3 & 9 \\ 1 & -3 \end{pmatrix}} x$

$\vec{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \vec{x}$, $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{pmatrix}$

eigenvalues:

$\begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (3+\lambda)(1+\lambda) + 2$

$= 3 + 4\lambda + \lambda^2 + 2 = 0$

$\lambda^2 + 4\lambda + 5 = 0$

$\lambda_1 = \dfrac{-4 + \sqrt{16-20}}{2} = -2 + i$, $\lambda_2 = -2 - i$

Corresponding eigenvector: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$,

$(A - (-2+i)I)\vec{v} = \vec{0}$

$\begin{pmatrix} -3-(-2+i) & 2 \\ -1 & -1-(-2+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{cases} -(1+i)v_1 + 2v_2 = 0 \\ -v_1 + (1-i)v_2 = 0 \end{cases} \Rightarrow v_1 = (1-i)v_2$

$\vec{v} = \begin{pmatrix} (1-i)v_2 \\ v_2 \end{pmatrix} \underset{=}{\overset{v_2=1}{=}} \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

solutions:

$$\vec{v} e^{\lambda_1 t} = \left[\begin{pmatrix}1\\1\end{pmatrix} + i\begin{pmatrix}-1\\0\end{pmatrix}\right] e^{(-2+i)t}$$

$$\boxed{e^{(-2+i)t} = e^{-2t}(\cos t + i\sin t)}$$

$$= \left[\begin{pmatrix}1\\1\end{pmatrix} + i\begin{pmatrix}-1\\0\end{pmatrix}\right] e^{-2t}(\cos t + i\sin t)$$

$$= e^{-2t}\left[\begin{pmatrix}1\\1\end{pmatrix}\cos t + i\begin{pmatrix}1\\1\end{pmatrix}\sin t + i\begin{pmatrix}-1\\0\end{pmatrix}\cos t + i^2\begin{pmatrix}-1\\0\end{pmatrix}\sin t\right]$$

$$= e^{-2t}\left\{\begin{pmatrix}1\\1\end{pmatrix}\cos t - \begin{pmatrix}-1\\0\end{pmatrix}\sin t + i\left[\begin{pmatrix}1\\1\end{pmatrix}\sin t + \begin{pmatrix}-1\\0\end{pmatrix}\cos t\right]\right\}$$

$$= e^{-2t}\left[\begin{pmatrix}\cos t+\sin t\\ \cos t\end{pmatrix} + i\begin{pmatrix}\sin t - \cos t\\ \sin t\end{pmatrix}\right]$$

General solution $\boxed{\vec{x}(t) = \left[c_1 \begin{pmatrix}\cos t + \sin t\\ \cos t\end{pmatrix} + c_2\begin{pmatrix}\sin t-\cos t\\ \sin t\end{pmatrix}\right]e^{-2t}}$