Due Thursday, April 6 at the beginning of class.
If you use convolutions, please write your answer in terms of convolution integrals.

1. Find the Laplace transform of the given function

   (a) $f(t) = \begin{cases} 0, & 0 \le t < 1, \\ 6t - 5, & 1 \le t < 3, \\ t^2, & t \ge 3. \end{cases}$

   (b) $f(t) = \displaystyle\int_0^t e^{-(t-\tau)} \sin\tau \, d\tau$

2. Find the inverse Laplace transform of

   (a) $\dfrac{e^{-2s} - 3e^{-7s}}{(s+5)^2}$

   (b) $\dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$

   (c) $\dfrac{1}{(s+1)^2(s^2+4)}$

3. Solve the initial value problem using the method of Laplace transform.

   (a) $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$, $g(t) = \begin{cases} t/2, & 0 \le t < 6, \\ 3, & t \ge 6. \end{cases}$

   (b) $y'' + y' + \dfrac{5}{4}y = t - u_{\pi/2}(t)\left(t - \dfrac{\pi}{2}\right)$, $y(0) = y'(0) = 0$

   (c) $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi)$, $y(0) = y'(0) = 0$

4. Express the solution of the initial value problem

$$4y'' + 4y' + 17y = g(t), \quad y(0) = y'(0) = 0$$

   in terms of a convolution integral.

Solutions.

#1a). $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 6t-5, & 1 \le t < 3 \\ t^2, & t \geq 3 \end{cases} = 0 + (6t-5)u_1(t) + (t^2-6t+5)u_3(t)$

$= (6t-5)u_1(t) \quad t+5)u_3(t)$

$\mathcal{L}\{f\} = \mathcal{L}\{(6t-5)u_1(t) + (t^2-6t+5)u_3(t)\}$

$= \mathcal{L}\{(6t-6+1)u_1(t)\} + \mathcal{L}\{((t-3)^2-4)u_3(t)\}$

$= 6\mathcal{L}\{(t-1)u_1(t)\} + \mathcal{L}\{u_1(t)\} + \mathcal{L}\{(t)u_3(t)\} - 4\mathcal{L}\{u_3(t)\}$

$= 6e^{-s}\mathcal{L}\{t\} + \dfrac{e^{-s}}{s} + e^{-3s}\mathcal{L}\{t^2\} - 4\dfrac{e^{-3s}}{s}$

$= \boxed{6\dfrac{e^{-s}}{s^2} + \dfrac{e^{-s}}{s} + \dfrac{2e^{-3s}}{s^3} - 4\dfrac{e^{-3s}}{s}}$

#1b) $f(t)=\int_0^t e^{-(t-\tau)}\sin\tau\,d\tau=(g*h)(t)$, where

$g(t)=e^{-t}$, $h(t)=\sin t$.

$$\mathcal{L}\{f\}=\mathcal{L}\{g\}\mathcal{L}\{h\}=\boxed{\frac{1}{s+1}\cdot\frac{1}{s^2+1}}$$

#2a) $F(s) = \dfrac{e^{-2s} - 3e^{-7s}}{(s+5)^2}$

$$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s+5)^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{e^{-7s}}{(s+5)^2}\right\}$$

$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)\, f(t-c)$, where $F(s) = \mathcal{L}\{f(t)\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+5)^2}\right\} = te^{-5t} = f(t)$$

$$\mathcal{L}^{-1}\{F\} = u_2(t) f(t-2) - 3u_7(t) f(t-7)$$

$$= \boxed{u_2(t)(t-2)e^{-5(t-2)} - 3u_7(t)(t-7)e^{-5(t-7)}}$$

#26) $F(s) = \dfrac{(s-2)e^{-s}}{s^2-4s+3}$

Partial fractions:

$$\frac{s-2}{s^2-4s+3} = \frac{A}{s-3} + \frac{B}{s-1} = \frac{A(s-1)+B(s-3)}{(s-3)(s-1)}$$

$$s-2 = A(s-1)+B(s-3)$$

$s=1$: $\quad -1 = -2B \Rightarrow \quad B = 1/2$

$s=3$: $\quad 1 = 2A \Rightarrow \quad A = 1/2$

$$\frac{s-2}{s^2-4s+3} = \frac{1}{2}\,\frac{1}{s-3} + \frac{1}{2}\,\frac{1}{s-1}$$

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-4s+3}\right\} = \frac{1}{2}e^{3t} + \frac{1}{2}e^{t} = f(t)$$

$$\mathcal{L}^{-1}\left\{\frac{(s-2)e^{-s}}{s^2-4s+3}\right\} = u_1(t)\,f(t-1)$$

$$= \boxed{\frac{1}{2}\left[e^{3(t-1)} + e^{t-1}\right]u_1(t)}$$

#2c) $F(s)=\dfrac{1}{(s+1)^2(s^2+4)}=\dfrac{1}{(s+1)^2}\cdot\dfrac{1}{s^2+4}$

$g(t)=\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\}=te^{-t}$

$h(t)=\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\}=\dfrac{1}{2}\sin 2t$

$\mathcal{L}^{-1}\{F(s)\}=(g*h)(t)=\int_0^t g(\tau)\,h(t-\tau)\,d\tau$

$=\boxed{\dfrac{1}{2}\int_0^t \sin[2(t-\tau)]\,\tau e^{-\tau}\,d\tau}$

#3a) $y''+y=g(t)$, $y(0)=0$, $y'(0)=1$, $g(t)=\begin{cases}\frac{t}{2}, & 0\le t<6\\ 3, & t\ge 6\end{cases}$

$\mathcal{L}\{y''+y\}=\mathcal{L}\{g(t)\}$

$\mathcal{L}\{y\}=Y(s)$

$\mathcal{L}\{y''\}=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)-1$

$\mathcal{L}\{g(t)\}=\mathcal{L}\{\frac{t}{2}+(3-\frac{t}{2})u_6(t)\}$

$=\frac{1}{2}\mathcal{L}\{t\}-\frac{1}{2}\mathcal{L}\{(t-6)u_6(t)\}=\frac{1}{2s^2}-\frac{1}{2}e^{-6s}\mathcal{L}\{t\}$

$=\frac{1}{2s^2}-\frac{1}{2}\frac{e^{-6s}}{s^2}$

$s^2Y(s)-1+Y(s)=\frac{1}{2s^2}-\frac{e^{-6s}}{2s^2}$

$Y(s)(s^2+1)=\frac{1}{2s^2}-\frac{e^{-6s}}{2s^2}+1$

$Y(s)=\frac{1}{2s^2(s^2+1)}-\frac{e^{-6s}}{2s^2(s^2+1)}+\frac{1}{s^2+1}$

$\frac{1}{s^2(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1}$

$\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+1)}\}=t-\sin t$

$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\boxed{\frac{1}{2}(t-\sin t)-\frac{1}{2}(t-6-\sin(t-6))u_6(t)+\sin t}$

#3b) $y''+y'+\frac{5}{4}y = t-u_{\pi/2}(t)(t-\frac{\pi}{2})$, $y(0)=y'(0)=0$

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = sY(s)-y(0) = sY(s)$

$\mathcal{L}\{y''\} = s^2Y(s)-sy(0)-y(1) = s^2Y(s)$

$\mathcal{L}\{t-u_{\pi/2}(t)(t-\pi/2)\} = \frac{1}{s^2} - e^{-\frac{\pi}{2}s}\mathcal{L}\{t\} = \frac{1}{s^2} - \frac{e^{-\pi/2 s}}{s^2}$

$s^2Y(s)+sY(s)+\frac{5}{4}Y(s) = \frac{1}{s^2} - \frac{e^{-\pi/2 s}}{s^2}$

$Y(s)(s^2+s+\frac{5}{4}) = \frac{1}{s^2} - \frac{e^{-\pi/2 s}}{s^2}$

$Y(s) = \frac{1}{s^2(s^2+s+\frac{5}{4})} - \frac{e^{-\pi/2 s}}{s^2(s^2+s+\frac{5}{4})}$

Partial fractions:

$$\frac{1}{s^2(s^2+s+\frac{5}{4})} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+s+\frac{5}{4}}$$

$$= \frac{As(s^2+s+5/4) + B(s^2+s+5/4) + (Cs+D)s^2}{s^2(s^2+s+5/4)}$$

$$= \frac{s^3(A+C) + s^2(A+B+D) + s(\frac{5}{4}A+B) + \frac{5}{4}B}{s^2(s^2+s+5/4)}$$

s^3: $0 = A + C$ | $C = -A = \frac{16}{25}$

s^2: $0 = A + B + D$ | $D = -A - B = -\frac{4}{5} + \frac{16}{25} = -\frac{4}{25}$

s: $0 = \frac{5}{4}A + B$ | $A = -\frac{4}{5}B = \frac{-16}{25}$

1: $1 = \frac{5}{4}B \Rightarrow$ | $B = \frac{4}{5}$

$$\frac{1}{s^2(s^2+s+5/4)} = -\frac{16}{25}\frac{1}{s} + \frac{4}{5}\frac{1}{s^2} + \frac{\frac{16}{25}s - \frac{4}{25}}{s^2+s+5/4}$$

$$\frac{\frac{16}{25}s - \frac{4}{25}}{s^2+s+5/4} = \frac{16}{25}\frac{s-\frac{1}{4}}{(s+\frac{1}{2})^2+1} = \frac{16}{25}\frac{s+\frac{1}{2}-\frac{3}{4}}{(s+\frac{1}{2})^2+1}$$

$$= \frac{16}{25}\frac{s+1/2}{(s+1/2)^2+1} - \frac{16}{25}\cdot\frac{3}{4}\frac{1}{(s+1/2)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+s+\frac{5}{4})}\right\} = -\frac{16}{25} + \frac{4}{5}t + \frac{16}{25}e^{-1/2t}\cos t - \frac{12}{25}e^{-1/2t}\sin t = f(t)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi/2 s}}{s^2(s^2+s+\frac{5}{4})}\right\} = u_{\frac{\pi}{2}}(t)\, f\left(t-\frac{\pi}{2}\right)$$

$$\boxed{\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= -\frac{16}{25} + \frac{4}{5}t + \frac{16}{25}e^{-1/2t}\cos t - \frac{12}{25}e^{-1/2t}\sin t \\ &-\left[-\frac{16}{25} + \frac{4}{5}\left(t-\frac{\pi}{2}\right) + \frac{16}{25}e^{-\frac{1}{2}(t-\frac{\pi}{2})}\underbrace{\cos\left(t-\frac{\pi}{2}\right)}_{\sin t} - \frac{12}{25}e^{-\frac{1}{2}(t-\pi/2)}\underbrace{\sin\left(t-\pi/2\right)}_{\cos t}\right]u_{\pi/2}(t) \end{aligned}}$$

#3c) $y''+4y=\delta(t-\pi)-\delta(t-2\pi),\ y(0)=y'(0)=0$

$\mathcal{L}\{y(t)\}=Y(s)$

$\mathcal{L}\{y''(t)\}=s^2Y(s)-sy(0)-y'(0)=s^2Y(s)$

$\mathcal{L}\{\delta(t-\pi)-\delta(t-2\pi)\}=e^{-\pi s}-e^{-2\pi s}$

$s^2Y(s)+4Y(s)=e^{-\pi s}-e^{-2\pi s}$

$Y(s)(s^2+1)=e^{-\pi s}-e^{-2\pi s}$

$Y(s)=\dfrac{e^{-\pi s}}{s^2+1}-\dfrac{e^{-2\pi s}}{s^2+1}$

$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\}-\mathcal{L}^{-1}\left\{\dfrac{e^{\cdot 2\pi s}}{s^2+1}\right\}$

$=u_\pi(t)\sin(t-\pi)-u_{2\pi}(t)\sin(t-2\pi)$

$=\boxed{-u_\pi(t)\sin t-u_{2\pi}(t)\sin t}$

#4.

$4y'' + 4y' + 17y = g(t),\ y(0) = y'(0) = 0$

$\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$

$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$

$\mathcal{L}\{g(t)\} = G(s)$

$(4s^2+4s+17)Y(s) = G(s)$

$$Y(s) = \frac{G(s)}{4s^2+4s+17} = \frac{1}{4}\,G(s)\,\frac{1}{s^2+s+17/4}$$

$\mathcal{L}^{-1}\{G(s)\} = g(t)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+17/4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1/2)^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1/2)^2+2^2}\right\}$$

$$= \frac{1}{2}e^{-t/2}\sin t = h(t)$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{8}(g*h)(t) = \frac{1}{8}\int_0^t g(t-\tau)h(\tau)\,d\tau$$

$$= \boxed{\frac{1}{8}\int_0^t g(t-\tau)e^{-\tau/2}\sin\tau\,d\tau}$$