HW 9. Key.

1. (a) $y'' + 0.5y' + 2y = 3\sin t. \Rightarrow y'' = 3\sin t - 0.5y' - 2y$

$y = x_1$

$y' = x_2$

$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = 3\sin t - 0.5x_2 - 2x_1 \end{cases}}$$

(b) $y'' + 0.25y' + 4y = 2\cos 3t, \quad y(0) = 1, y'(0) = -2$

$y = x_1$

$y' = x_2$

$y'' = 2\cos 3t - 0.25y' - 4y.$

$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = 2\cos 3t - 0.25x_2 - 4x_1 \end{cases} \quad x_1(0) = 1, \; x_2(0) = -2}$$

2. $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}, \; B = \begin{pmatrix} i & 3 \\ 2 & 2i \end{pmatrix}$

(a) $\cancel{2A - 3B = \begin{pmatrix} 2(1+i) - 3i & 2(-1+2i) - 9 \\ 2(3+2i) - 2(3) & 2(2-i) - 6i \end{pmatrix} = \begin{pmatrix} 2-i & -11+2i \\ 2i & 4-8i \end{pmatrix}}$

$3A - 2B = \begin{pmatrix} 3(1+i) - 2i & 3(-1+2i) - 6 \\ 3(3+2i) - 4 & 3(2-i) - 4i \end{pmatrix} = \begin{pmatrix} 3+i & -9+6i \\ 5+6i & 6-7i \end{pmatrix}$

(b) $$AB = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}\begin{pmatrix} i & 3 \\ 2 & 2i \end{pmatrix}$$

$$= \begin{pmatrix} (1+i)i + (-1+2i)(2) & (1+i)3 + (-1+2i)(2i) \\ (3+2i)i + (2-i)(2) & (3+2i)3 + (2-i)(2i) \end{pmatrix}$$

$$= \begin{pmatrix} i+i^2-2+4i & 3+3i-2i+4i^2 \\ 3i+2i^2+4-2i & 9+6i+4i-2i^2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 5i-3 & -1+i \\ i+2 & 11+10i \end{pmatrix}}$$

(c) $$BA = \begin{pmatrix} i & 3 \\ 2 & 2i \end{pmatrix}\begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$$

$$= \begin{pmatrix} i(1+i)+3(3+2i) & i(-1+2i)+3(2-i) \\ 2(1+i)+2i(3+2i) & 2(-1+2i)+2i(2-i) \end{pmatrix}$$

$$= \begin{pmatrix} i+i^2+9+6i & -i+2i^2+6-3i \\ 2+2i+6i+4i^2 & -2+4i+4i-2i^2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 8+7i & 4-4i \\ -2+8i & 8i \end{pmatrix}}$$

3. $$A = \begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}, \quad \det A = 3+8 = 11$$

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 3 & -4 \\ 2 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{3}{11} & -\frac{4}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}}$$

4. $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

$$\det A = \begin{vmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 3 & 2 \end{vmatrix} \begin{matrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 3 & 2 \end{matrix}$$

$$= 0 + 16 + 24 - 0 - 8 - 18 = \boxed{14}$$ $\det A = 8$

5. (a) $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix}$

Eigenvalues: $\cancel{\begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda}$

$\det A = -4 + 6 = 2.$

$\text{trace}(A) = 1 - 4 = -3.$

Characteristic equation

$$\lambda^2 + 3\lambda + 2 = 0.$$

$$(\lambda+2)(\lambda+1) = 0.$$

$$\boxed{\lambda_1 = -1, \ \lambda_2 = -2 \text{ eigenvalues}}$$

<u>Eigenvectors:</u>

$\lambda_1 = -1$: $A - (-1)I = \begin{pmatrix} 1+1 & -2 \\ 3 & -4+1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2v_1 - 2v_2 = 0 \\ 3v_1 - 3v_2 = 0 \end{cases} \Rightarrow v_1 = v_2 \Rightarrow v = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ - eigenvector corresponding } \lambda_1 = -1}$$

$\lambda_2 = -2.$

$$A-(-2)I = \begin{pmatrix} 1-(-2) & -2 \\ 3 & -4(-2) \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}$$

$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\begin{cases} 3u_1 - 2u_2 = 0 \\ 3u_1 - 2u_2 = 0 \end{cases}$ $\qquad u_1 = +\frac{2}{3}u_2$

$$\vec{u} = \begin{pmatrix} +\frac{2}{3}u_2 \\ u_2 \end{pmatrix} = u_2 \begin{pmatrix} +\frac{2}{3} \\ 1 \end{pmatrix}$$

$\boxed{\vec{x}_2 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}}$ - eigenvector corresponding to $\lambda_2 = -2$

or $\quad u_2 = \dfrac{3u_1}{2} \Rightarrow \vec{u} = u_1 \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

$$\boxed{\vec{x}_2 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}}.$$

(1b) $\quad B = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}, \quad B - \lambda I = \begin{pmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix}$

det $B = -4+5 = 1$

trace $B = 2-2 = 0.$

characteristic equation: $\lambda^2 + 1 = 0$

$\boxed{\lambda = \pm i.}$ eigenvalues

Eigenvectors:

$\lambda_1 = i.$

$$B - iI = \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (2-i)v_1 - 5v_2 = 0 \Rightarrow \cancel{v_2 = \frac{2-i}{5} v_1} \\ v_1 - (-2-i)v_2 = 0 \Rightarrow v_1 = (-2-i)v_2. \end{cases}$$

$$\boxed{\vec{v} = \begin{pmatrix} (+2-i)v_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} +2-i \\ 1 \end{pmatrix} \text{ - eigenvector corresponding to } \lambda = i}$$

$$\boxed{\begin{gathered} \lambda_2 = -i \\ \vec{u} = (\overline{\vec{v}}) = \begin{pmatrix} +2+i \\ 1 \end{pmatrix} \text{ - eigenvector corresponding to } \lambda = -i \end{gathered}}$$

(c) $C = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, $C - \lambda I = \begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix}$

$\det C = -4 + 3 = -1$

trace $C = 0$

Characteristic equation:

$$\lambda^2 - 1 = 0.$$

$$\boxed{\lambda = \pm 1. \text{ - eigenvalues}}$$

<u>Eigenvectors:</u>

$\lambda_1 = 1.$ $\quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$(C - I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} v_1 - v_2 = 0 \\ 3v_1 - 3v_2 = 0 \end{cases} \Rightarrow v_1 = v_2$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \boxed{\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\lambda_2 = -1. \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$(A+I)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3u_1 - u_2 = 0 \\ 3u_1 - u_2 = 0 \end{cases} \Rightarrow u_2 = 3u_1$$

$$\vec{u} = \begin{pmatrix} \cancel{u_2}\ u_1 \\ 3u_1 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\boxed{\vec{x}_2 = \begin{pmatrix} \cancel{1}\ 1 \\ \cancel{3}\ 3 \end{pmatrix}}$$