6. $e^{2+\frac{3\pi}{4}i} =$

(a) π

(b) $\dfrac{\sqrt{2}}{2}(1+i)e^2$

(c) $\dfrac{\sqrt{2}}{2}(1-i)e^2$

(d) $\dfrac{\sqrt{2}}{2}(-1+i)e^2$

Euler's formula

$$e^{a+bi} = e^a(\cos b + i \sin b)$$

$$e^{2+\frac{3\pi}{4}i} = e^2\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$$

$$= e^2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

11. Find a general solution to the equation

$$y'' + 6y' + 9y = \frac{e^{-3x}}{1+2x}$$

variation of parameters.

homogeneous eqn:

$$y''+6y'+9y=0$$
$$r^2+6r+9=0$$
$$(r+3)^2=0$$
$$r=-3 \text{ - repeated root}$$

$$y_h(x)=(C_1+C_2x)e^{-3x}$$
$$=C_1\underbrace{e^{-3x}}_{y_1(x)}+C_2\underbrace{xe^{-3x}}_{y_2(x)}$$

$$y(x)=C_1(x)e^{-3x}+C_2(x)xe^{-3x}$$

$$\begin{cases} C_1'y_1+C_2'y_2=0 \\ C_1'y_1'+C_2'y_2'=g(x)\end{cases}$$

$$\begin{cases} C_1'\cancel{e^{-3x}}+C_2'x\cancel{e^{-3x}}=0 \\ C_1'(-3)\cancel{e^{-3x}}+C_2'\cancel{e^{-3x}}+C_2'x(-3)\cancel{e^{-3x}}=\frac{\cancel{e^{-3x}}}{1+2x}\end{cases}$$

$$\begin{cases} C_1'+C_2'x=0 \Rightarrow \boxed{C_1'=-C_2'x} \\ -3C_1'-3xC_2'+C_2'=\frac{1}{1+2x}\end{cases}$$

$$C_2'=\frac{1}{1+2x}$$
$$\boxed{C_2(x)=\frac{1}{2}\ln|1+2x|+C_3}$$

$$C_1'(x)=-xC_2'$$
$$=-\frac{x}{1+2x} \text{ - improper fraction}$$

$$\boxed{\begin{array}{r} \tfrac{1}{2} \\ x\,\big)\overline{1+2x} \\ -x+\tfrac{1}{2} \\ \hline -\tfrac{1}{2} \end{array}}$$

$$=-\left(\frac{1}{2}-\frac{1}{2}\frac{1}{1+2x}\right)$$
$$C_1'(x)=-\frac{1}{2}+\frac{1}{2+4x}$$
$$\boxed{C_1(x)=-\frac{1}{2}x+\frac{1}{4}\ln|2+4x|+C_4}$$

$$\boxed{y(x)=\left(-\frac{1}{2}x+\frac{1}{4}\ln|2+4x|+C_4\right)e^{-3x}+\left(\frac{1}{2}\ln|1+2x|+C_3\right)xe^{-3x}}$$

general solution.

12. Find a general solution to the equation

$$4y'' + y' = 4x^3 + 48x^2 + 1$$

undetermined coefficients.

homogeneous eqn.

$4y'' + y' = 0$

$4r^2 + r = 0$

$r(4r+1) = 0$

$r_1 = 0, \quad r_2 = -1/4$

$y_h(x) = C_1 + C_2 e^{-\frac{1}{4}x}$

r=0 is a root of the auxiliary eqn

$y_p(x) = x(Ax^3 + Bx^2 + Cx + D)$

$= Ax^4 + Bx^3 + Cx^2 + Dx$

$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D$

$y_p'' = 12Ax^2 + 6Bx + 2C$

$4(12Ax^2 + 6Bx + 2C) + 4Ax^3 + 3Bx^2 + 2Cx + D = 4x^3 + 48x^2 + 1$

$x^3: \quad 4A = 4 \qquad \boxed{A=1}$

$x^2: \quad 48A + 3B = 48 \qquad 3B = 48 - 48A = 0 \qquad \boxed{B=0}$

$x: \quad 24B + 2C = 0 \qquad \boxed{C=0}$

$1: \quad 8C + D = 1 \qquad \boxed{D=1}$

$y_p(x) = x^4 + x$

$$\boxed{y(t) = C_1 + C_2 e^{-\frac{1}{4}x} + x^4 + x}$$

13. Given that $y_1(x) = x$ is a solution to

$$x^2y'' + xy' - y = 0,$$

find a second solution of this equation on $(0, +\infty)$.

reduction of order

General solution: $y(x) = y_1(x)\underbrace{v(x)}_{\text{unknown function}}$

$= x\,v(x)$

$y'(x) = v(x) + xv'(x)$

$y''(x) = v'(x) + v'(x) + xv''(x)$

$= 2v'(x) + xv''(x)$

Plug y, y', y'' into the eqn:

$x^2\left(2v'(x) + xv''(x)\right) + x\left(v(x) + xv'(x)\right) - xv(x) = 0$

$2x^2v'(x) + x^3v''(x) + \cancel{xv(x)} + x^2v'(x) - \cancel{xv(x)} = 0$

$\dfrac{x^3v''(x) + 3x^2v'(x) = 0}{x^2}$

$xv''(x) + 3v'(x) = 0$

Substitution: $v'(x) = u(x)$

$v''(x) = u'(x)$

$xu' + 3u = 0$ $\qquad x\dfrac{du}{dx} + 3u = 0$

$\displaystyle\int \frac{du}{u} = -3\int\frac{dx}{x}$

$\ln|u| = -3\ln|x| + \ln C$

$u(x) = C_1x^{-3}$

$v'(x) = C_1x^{-3}$

$v(x) = C_1\dfrac{x^{-2}}{-2} + C_2$

$= C_3x^{-2} + C_2$ $\qquad \boxed{C_3 = \dfrac{C_1}{-2}}$

$y(x) = (C_3x^{-2} + C_2)x$

$= C_3\underbrace{x^{-1}}_{y_2(x)} + C_2\underbrace{x}_{y_1(x)}$

$\boxed{y_2(x) = x^{-1}}$

Title : Mar 18-9:47 AM (Page 4 of 4)