MATH 308 – 530, 531 **Practice Problems for Test 2** **Spring 2016**

over sections 3.3 – 3.8, 6.1

1. $e^{2+\frac{3\pi}{4}i} =$

2. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in. then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period and amplitude of the motion.

3. A mass weighting 8 lb is attached to a spring hanging from the ceiling and comes to rest at its equilibrium position. At $t = 0$, an external force $F(t) = 2\cos 2t$ lb is applied to the system. If the spring constant is 10 lb/ft and the damping constant is 1 lb-sec/ft, find the steady-state solution for the system. What is the resonance force for the system?

4. Find the general solution of the equation

   (a) $y'' - 2y' + 5y = 0$

   (b) $y'' + 6y' + 9y = \frac{e^{-3x}}{1+2x}$

   (c) $4y'' + y' = 4x^3 + 48x^2 + 1$

5. Given that $y_1(x) = x$ is a solution to

$$x^2y'' + xy' - y = 0,$$

   find a second solution of this equation on $(0, +\infty)$.

6. Find the Laplace transform of the given function using the definition of the Laplace transform.

   (a) $f(\cancel{x}\,t) = te^{3t}$.

   (b) $f(t) = \begin{cases} e^{5t} & 0 \le t < 6 \\ 3 & t \ge 6. \end{cases}$

7. Find the Laplace transform of

   (a) $f(t) = t\cos 3t$

   (b) $f(t) = t^2e^{-2t}$

8. Find the inverse Laplace transform of the given function.

   (a) $F(s) = \dfrac{2s+6}{s^2-4s+8}$

   (b) $F(s) = \dfrac{1\,\cancel{e^{-2s}}}{s^2+s-2}$

#6.

a) $\mathcal{L}\{te^{3t}\} = \int_0^\infty te^{3t}e^{-st}\,dt = \int_0^\infty te^{(3-s)t}\,dt = \lim_{N\to\infty}\int_0^N t\,e^{(3-s)t}\,dt$

Integrate by parts: $u = t$ $\quad du = dt$

$dv = e^{(3-s)t}\,dt$ $\quad v = \frac{1}{3-s}e^{(3-s)t}$

$$= \lim_{N\to\infty} \left\{ \frac{t}{3-s} e^{(3-s)t} \Big]_0^N - \frac{1}{3-s}\int_0^N e^{(3-s)t}\,dt \right\}$$

$$= \lim_{N\to\infty} \left\{ \frac{N}{3-s} e^{(3-s)N} - \frac{1}{(3-s)^2} e^{(3-s)t} \Big]_0^N \right\}$$

$$= \lim_{N\to\infty} \left\{ \frac{N}{(3-s)e^{(s-3)N}} - \frac{1}{(3-s)^2} \underbrace{e^{(3-s)N}}_{0 \text{ if } s>3} + \frac{1}{(3-s)^2} \right\}$$

$$= \lim_{N\to\infty} \left\{ \underbrace{\frac{1}{(3-s)(s-3)e^{(s-3)N}}}_{0 \text{ if } s>3} \right\} + \frac{1}{(3-s)^2} = \frac{1}{(3-s)^2} = \boxed{\frac{1}{(s-3)^2}}, \quad s>3$$

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b) $\mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st}\,dt = \int_0^6 e^{5t}e^{-st}\,dt + \int_6^\infty 3e^{-st}\,dt$

$$= \int_0^6 e^{(5-s)t}\,dt + 3\int_6^\infty e^{-st}\,dt = \frac{1}{5-s} e^{(5-s)t}\Big]_0^6 - \frac{3}{s}e^{-st}\Big]_6^\infty$$

$$= \boxed{\frac{1}{5-s}\left(e^{6(5-s)}-1\right) + \frac{3}{s}e^{-6s}}$$

#7.

(a) $\mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9}$

$$\mathcal{L}\{t\cos 3t\} = -\left(\frac{s}{s^2+9}\right)' = -\frac{s^2+9-2s(s)}{(s^2+9)^2} = -\frac{9-s^2}{(s^2+9)^2} = \boxed{\frac{s^2-9}{(s^2+9)^2}}$$

(b) $\mathcal{L}\{t^2e^{-2t}\}$

$\mathcal{L}\{f(t)e^{at}\} = F(s-a)$, where $F(s) = \mathcal{L}\{f\}$

$\mathcal{L}\{t^2\} = \frac{2}{s^3}$

$\mathcal{L}\{t^2e^{-2t}\} = \boxed{\frac{2}{(s+2)^3}}$

#8.

(a) $F(s) = \frac{2s+6}{s^2-4s+8} = 2\frac{s+3}{(s-2)^2+2^2} = 2\frac{(s-2)+5}{(s-2)^2+2^2} = 2\left[\frac{s-2}{(s-2)^2+2^2} + \frac{5}{2}\frac{2}{(s-2)^2+4}\right]$

$\mathcal{L}^{-1}\{F\} = 2\left[e^{2t}\cos 2t + \frac{5}{2}e^{2t}\sin 2t\right] = \boxed{e^{2t}(2\cos 2t + 5\sin 2t)}$

(b) $\frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$

$1 = A(s-1)+B(s+2)$

$s=1: \quad 1 = 3B \Rightarrow B = 1/3$

$s=-2: \quad 1 = -3A \Rightarrow A = -1/3$

$$\frac{1}{s^2+s-2} = -\frac{1}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}$$

$$\mathcal{L}^{-1}\{F(s)\} = \boxed{\frac{-1}{3}e^{-2t} + \frac{1}{3}e^{t}}$$

4a) $y''-2y'+5y=0$

$r^2-2r+5=0$

$r_{1,2} = \frac{2 \pm \sqrt{16-20}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i \qquad Re(1+i) = 1 \qquad Im(1+i) = 1$

General solution

$y(t) = (c_1 \cos t + c_2 \sin t)e^{t}$