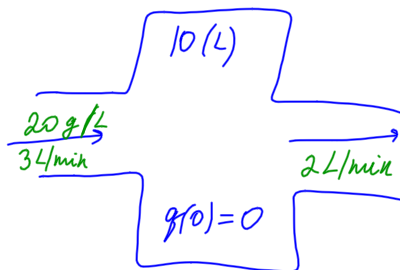


1. A large tank initially contains 10 L of fresh water. A brine containing 20 g/L of salt flows into the tank at a rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the rate of 2 L/min. Determine the concentration of salt in the tank as a function of time.



$g(t)$  is the mass of salt in the tank at time  $t$ .

$$\frac{dg}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = 3(20)$$

$$\text{rate out} = \frac{g(t)}{10+(3-2)t} \cdot 2$$

$$\frac{dg}{dt} = 60 - \frac{2g(t)}{10+t} \quad \text{IVP}$$

$$g(0) = 0$$

Solve for  $g(t)$ .

So  $\frac{g(t)}{10+t}$  concentration

2. Suppose that a sum  $S_0$  is invested at an annual rate of return  $r$  compounded continuously.

(a) Find the time  $T$  required for the original sum to double in value as a function of  $r$ .

$S(t)$  is the balance at time  $t$ .

$$\frac{dS}{dt} = rS$$

$$S(0) = S_0$$

Find  $T$  such that  $S(T) = 2S_0$

---

$$\frac{dS}{S} = r dt$$

$$\ln|S| = rt + C$$

$$S(t) = Ce^{rt}$$

$$S_0 = S(0) = C$$

$$S(t) = S_0 e^{rt}$$

---

$$S(T) = S_0 e^{rT} = 2S_0$$

$$T = \frac{\ln 2}{r}$$

(b) Determine  $T$  if  $r = 7\% = 0.07$

$$T = \frac{\ln 2}{0.07} \approx \boxed{10 \text{ (years)}}$$

(c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

$$8 = \frac{\ln 2}{r}$$

$$r = \frac{\ln 2}{8} \approx 0.09 \quad \boxed{9\%}$$

3. An object with temperature  $150^{\circ}$  is placed in a freezer whose temperature is  $30^{\circ}$ . Assume that the temperature of the freezer remains essentially constant.

(a) If the object is cooled to  $120^{\circ}$  after 8 min, what will its temperature be after 18 min?

$T(t)$  - temperature at time  $t$

$$\frac{dT}{dt} = k(30 - T) \quad T(0) = 150$$

$$T(8) = 120$$

$$\text{Find } T(18) = ?$$

$$\frac{dT}{dt} = -k(T - 30)$$

$$\frac{dT}{T - 30} = -k dt$$

$$\ln |T - 30| = -kt + C$$

$$T - 30 = Ce^{-kt}$$

$$T = 30 + Ce^{-kt}$$

$$30 + C = 150$$

$$C = 120$$

$$T(t) = 30 + 120e^{-kt}$$

solve for  $k$

$$T(8) = 30 + 120e^{-8k} = 120$$

$$e^{-8k} = \frac{3}{4}$$

$$k = -\frac{1}{8} \ln\left(\frac{3}{4}\right) \approx 0.036$$

$$T(t) = 30 + 120e^{-0.036t}$$

$$T(18) = \dots$$

(b) When will its temperature be  $60^{\circ}$ ?

find  $t$  such that  $T(t) = 60$

4. Determine (without solving the problem) an interval in which the solution to the initial value problem

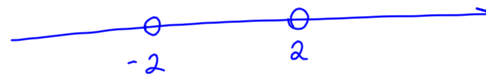
$$(4 - t^2)y' + 2ty = 3t^2, \quad y(1) = -3$$

is certain to exist.

$$(-2, 2)$$

$$y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$$

continuous if  $t \neq \pm 2$



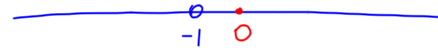
5. Solve the initial value problem

$$y' = \frac{t^2}{1+t^3}, \quad y(0) = y_0$$

and determine how the interval in which the solution exists depends on the initial value  $y_0$ .

$$f(t, y) = \frac{t^2}{1+t^3} \quad \text{continuous if } t \neq -1$$

$$\frac{\partial f}{\partial y} = 0$$



$$y' = \frac{t^2}{1+t^3}, \quad y(t_0) = y_0$$

$$\text{if } t_0 > -1, \quad (-1, \infty)$$

$$\text{if } t_0 < -1, \quad (-\infty, -1)$$

$$0 > -1, \text{ so } (-1, \infty)$$

does not depend on  $y_0$

Solve the initial value problem:

$$\frac{dy}{dt} = \frac{t^2}{1+t^3}$$

$$dy = \frac{t^2}{1+t^3} dt$$

$$y(t) = \frac{1}{3} \ln|1+t^3| + C$$

$$y(0) = C = y_0$$

$$y(t) = \frac{1}{3} \ln|1+t^3| + y_0 \quad \text{exists everywhere except } t = -1.$$

6. Solve the following initial value problem

$$\sqrt{y}dt + (1+t)dy = 0 \quad y(0) = 1.$$



7. Find the general solution to the equation

$$(t^2 - 1)y' + 2ty + 3 = 0$$

$$y' + \frac{2t}{t^2-1}y = -\frac{3}{t^2-1}$$

$$p(t) = \frac{2t}{t^2-1}; \quad q(t) = -\frac{3}{t^2-1}$$

Integrating factor:

$$\frac{d\mu(t)}{dt} = + \frac{2t\mu}{t^2-1}$$

$$\frac{d\mu}{\mu} = \frac{2t}{t^2-1} dt$$

$$\mu(t) = t^2 - 1$$

$$(t^2 - 1)y(t) = \int -\frac{3}{t^2-1} (t^2-1) dt$$

$$= -3t + C$$

$$y(t) = -\frac{3t}{t^2-1} + \frac{C}{t^2-1}$$

8. Solve the initial value problem

$$(uvw)' = u'vw + uv'w + uvw'$$

$$(ye^{xy} \cos(2x) - 2e^{xy} \sin(2x) + 2x)dx + (xe^{xy} \cos(2x) - 3)dy = 0, \quad y(0) = -1$$

$$M(x,y)$$

$$N(x,y)$$

$$M_y = e^{xy} \cos(2x) + xy e^{xy} \cos(2x) - 2x e^{xy} \sin(2x)$$

$$N_x = e^{xy} \cos(2x) + xy e^{xy} \cos(2x) - 2 \sin(2x) x e^{xy}$$

Exact

$$\left\{ \begin{array}{l} \frac{\partial F}{\partial x} = ye^{xy} \cos(2x) - 2e^{xy} \sin(2x) + 2x \\ \frac{\partial F}{\partial y} = (xe^{xy} \cos(2x) - 3)dy \end{array} \right.$$

$$\int e^{xy} dy = \frac{1}{x} e^{xy} + C$$

$$F(x,y) = x \frac{1}{x} e^{xy} \cos(2x) - 3y + h(x)$$

$$= e^{xy} \cos(2x) - 3y + h(x)$$

$$\frac{\partial F}{\partial x} = ye^{xy} \cos(2x) - 2e^{xy} \sin(2x) + h'(x)$$

$$h'(x) = 2x \quad \text{or} \quad h(x) = x^2 + C$$

$$F(x,y) = e^{xy} \cos(2x) - 3y + x^2 + C$$

$$\text{General solution: } e^{xy} \cos(2x) - 3y + x^2 + C = 0$$

$y(0) = -1$ . Plug  $y = -1$  and  $x = 0$  into the general solution:

$$e^{(0)(-1)} \cos(0) - 3(-1) + 0 + C = 0$$

$$C = -4$$

$$e^{xy} \cos(2x) - 3y + x^2 - 4 = 0$$

9. Find an integrating factor for the equation

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

$$\underbrace{(3xy + y^2)}_{M(x,y)} + \underbrace{(x^2 + xy)}_{N(x,y)}y' = 0$$

and then solve the equation.

$$\frac{M_y - N_x}{N(x,y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy}$$

$$= \frac{x + y}{x(x + y)}$$

$$= \frac{1}{x} \text{ (depends on } x \text{ only)}$$

Integrating factor:

$$\frac{d\mu}{dx} = \frac{\mu}{x}$$

$$\frac{d\mu}{\mu} = \frac{dx}{x}$$

$$\mu(x) = x$$

$$\underbrace{(3x^2y + xy^2)}_{M(x,y)}dx + \underbrace{(x^3 + x^2y)}_{N(x,y)}dy = 0$$

$$M_y = 3x^2 + 2xy$$

$$N_x = 3x^2 + 2xy$$

Exact

$$\left\{ \begin{array}{l} \frac{\partial F}{\partial x} = 3x^2y + xy^2 \\ \frac{\partial F}{\partial y} = x^3 + x^2y \end{array} \right.$$

$$\int \frac{\partial F}{\partial y} dy = \int (x^3 + x^2y) dy$$

$$F(x,y) = x^3y + \frac{x^2y^2}{2} + h(x)$$

$$\frac{\partial F}{\partial x} = 3x^2y + xy^2 + h'(x)$$

$$h'(x) = 0 \text{ or } h(x) = C$$

$$F(x,y) = x^3y + \frac{x^2y^2}{2} + C$$

General solution:

$$x^3y + \frac{x^2y^2}{2} + C = 0 \quad y=0$$

10. Solve the initial value problem

$$6y'' - 5y' + y = 0, \quad y(0) = 4, y'(0) = 0$$

11. Find the general solution to the equation

$$4y'' - 12y' + 9y = 0$$

$$4r^2 - 12r + 9 = 0$$

$$(2r - 3)^2 = 0$$

$$r = \frac{3}{2} \text{ repeated root}$$

General solution:

$$y(t) = (c_1 + c_2 t) e^{\frac{3}{2}t}$$

WIR 1

2. Given the differential equation

$$\frac{dy}{dt} = ty - 1.$$

- (a) What is the slope of the graph of the solutions at (0,1), at the point (1,1), at the point (3,-1), at the point (0,0)?
- (b) Find all the points where the tangents to the solution curves are horizontal?

#2a      slope @  $(t,y) = ty - 1$

            @  $(0,1) = -1$

            @  $(1,1) = 0$

            @  $(3,-1) = -4$

            @  $(0,0) = -1$

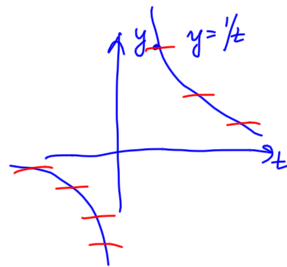
---

#2b      slope = 0

$ty - 1 = 0$

$$y = \frac{1}{t}$$

The slope is horizontal at all points on the hyperbola  $y = \frac{1}{t}$



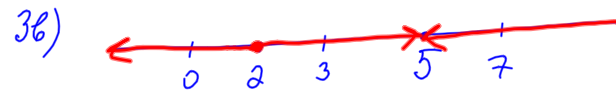
HW 3 #3

Given the differential equation

$$\frac{dy}{dt} = 7y - y^2 - 10$$

- (a) Find the equilibrium solutions.  
 (b) Sketch the phase line and determine whether the equilibrium solutions are stable, unstable, or semistable.

3a)  $7y - y^2 - 10 = 0$   
 $y^2 - 7y + 10 = 0$   
 $(y-5)(y-2) = 0$   
 $y_1 = 2$  *unstable*  
 $y_2 = 5$  *stable*



$$f(0) = -10 < 0$$

$$f(3) = 21 - 9 - 10 > 0$$

$$f(7) = 49 - 49 - 10 < 0$$