Section 2.1 **Linear Equations.**

Definition. A linear first-order equation is an equation that can be expressed in the form

$$\boxed{a_1(x)\frac{dy}{dx} + a_0(x)y = b(x),}$$

If $b(x)=0$ ($Q(x)=0$), then $a_1(x)y' + a_0(x)y = 0$ is separable

where $a_0(x)$, $a_1(x)$, $b(x)$ depend only on x.

We will assume that $a_0(x)$, $a_1(x)$, $b(x)$ are continuous functions of x on an interval I.

For now, we are interested in those linear equations for which $a_1(x)$ is never zero on I. In that case we can rewrite the equation in the standard form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x),}$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$ are continuous on I.

To solve the first order linear equation: ($Q(x) \neq 0$)

Find the integrating factor $\mu(x)$ such that $\frac{d}{dx}[\mu y] = \mu Q$

1. Write the equation $a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$ in the standart form $\frac{dy}{dx} + P(x)y = Q(x)$.
2. Find the integrating factor $\mu(x)$ solving differential equation

$$\boxed{\frac{d\mu}{dx} - P(x)\mu = 0.}$$

separable!

$\frac{dy}{dx} + P(x)y = \cancel{Q(x)}^{0}$

3. Integrate the equation

$$\int \frac{d}{dx}[\mu y]\,dx = \int \mu Q(x)\,dx$$

$$\mu y = \int \mu Q\,dx$$

and solve for y by dividing by $\mu(x)$.

The solution

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right]$$

where C is a constant, to the equation is called the **general solution**.

Example 1. Find the general solution to the following equations

1. $\underline{xy' - y = -\ln x}$, $x>0$
x

standart form: $y' - \frac{1}{x}y = -\frac{\ln x}{x}$ $\left(P(x) = -\frac{1}{x},\ Q(x) = -\frac{\ln x}{x}\right)$

Integrating factor $\mu(x)$: $\mu' + \frac{1}{x}\mu = 0$

$$\frac{d\mu}{dx} + \frac{1}{x}\mu = 0$$

$$\frac{d\mu}{\mu} = -\frac{dx}{x}$$

$$\ln|\mu| = -\ln|x|$$

$$\ln|\mu| = \ln\left|\frac{1}{x}\right|$$

$$\mu(x) = \frac{1}{x}$$

$$\frac{d}{dx}[\mu y] = \mu Q$$

$$\frac{d}{dx}\left[\frac{y}{x}\right] = -\frac{\ln x}{x^2}$$

$$\frac{y}{x} = -\int \frac{\ln x}{x^2}\,dx$$

by parts:
$u = \ln x$ $dv = \frac{dx}{x^2}$
$du = \frac{dx}{x}$ $v = -\frac{1}{x}$

$$\frac{y}{x} = -\left(\ln x\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\frac{dx}{x}\right)$$

$$\frac{y}{x} = \frac{\ln x}{x} - \int \frac{1}{x^2}\,dx$$

$$\frac{y}{x} = \frac{\ln x}{x} + \frac{1}{x} + C$$

solve for y: $\boxed{y = \ln x + 1 + Cx}$

2. $y' - 2y = t^2 e^{2t}$

Integrating factor $\mu(x)$:

$$\mu' + 2\mu = 0$$

$$\frac{d\mu}{dt} + 2\mu = 0$$

$$\int \frac{d\mu}{\mu} = -\int 2\,dt$$

$$\ln|\mu| = -2t$$

$$\boxed{\mu = e^{-2t}}$$

$$\frac{d}{dt}[\mu y] = \mu Q$$

$$\frac{d}{dt}[y e^{-2t}] = t^2 \overbrace{e^{2t} e^{-2t}}^{1}$$

$$\frac{d}{dt}[y e^{-2t}] = t^2$$

$$y e^{-2t} = \int t^2\,dt$$

$$y e^{-2t} = \frac{t^3}{3} + C$$

$$\boxed{y = \left(\frac{t^3}{3} + C\right) e^{2t}}$$

Example 2. Solve the initial value problem

1. $dy - y\,dx - 2xe^x\,dx = 0, \quad y(0) = e - 2$

$$\frac{dy}{dx} - y - 2xe^x = 0$$

$$\frac{dy}{dx} - y = 2xe^x$$

Integrating factor $\mu(x)$: $\frac{d\mu}{dx} + \mu = 0$

$$\int \frac{d\mu}{\mu} = -\int dx$$

$$\ln|\mu| = -x$$

$$\boxed{\mu = e^{-x}}$$

$$\frac{d}{dx}[\mu y] = \mu Q$$

$$\frac{d}{dx}[y e^{-x}] = 2xe^x e^{-x}$$

$$= 2x$$

$$ye^{-x} = x^2 + C$$

$$y(x) = (x^2 + c)e^x$$

Plug y into $y(0) = e-2$:

$$y(0) = C = e - 2$$

$$\boxed{y(x) = (x^2 + e - 2)e^x}$$

2. $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, t > 0$