Chapter 3. Second Order Linear Equations

Section 3.1 **Homogeneous Equations with Constant Coefficients**

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

where f is some given function.

An initial value problem consists of a differential equation together with the pair of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

A second order ordinary differential equation is said to be **linear** if it is written in the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

or

$$y'' + p(t)y' + q(t)y = g(t).$$

If $g(t) = 0$, then the equation is called **homogeneous**. Otherwise, the equation is called **nonhomogeneous**.

We begin our discussion with homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0$$

where a, b, c are constants.

We try to find a solution of the form $y = e^{rt}$.

Definition. An equation

$$ar^2 + br + c = 0$$

is called the **auxiliary equation** or **characteristic equation** associated with equation $ay'' + by' + cy = 0$.

If $b^2 - 4ac > 0$, then the auxiliary equation has two distinct real roots r_1 and r_2. Then

$$y(x) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is the general solution to the equation.

If $b^2 - 4ac = 0$, then the auxiliary equation has a repeated root $r \in \mathbf{R}$, $r = -\dfrac{b}{2a}$. In this case,

$$y(x) = c_1 e^{rx} + c_2 t e^{rt} = (c_1 + c_2 t)e^{rt}$$

is the general solution to the equation.

Example 1. Find the general solution to the given equation

(a) $y'' - y' - 2y = 0$.

(b) $y'' + 6y' + 9y = 0$.

Example 2. Solve the given initial value problems.

(a) $y'' + y' = 0$, $y(0) = 2$, $y'(0) = 1$.

(b) $y'' - 4y' + 4y = 0$, $y(1) = 1$, $y'(1) = 1$.