Chapter 3. Second Order Linear Equations

Section 3.1 **Homogeneous Equations with Constant Coefficients**

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

where f is some given function.

An initial value problem consists of a differential equation together with the pair of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

A second order ordinary differential equation is said to be linear if it is written in the form

$$\boxed{P(t)y'' + Q(t)y' + R(t)y = G(t)}$$

or

$$\boxed{y'' + p(t)y' + q(t)y = g(t).}$$

If $g(t) = 0$, then the equation is called homogeneous. Otherwise, the equation is called nonhomogeneous.

We begin our discussion with homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0$$

where a, b, c are constants.

We try to find a solution of the form $y = e^{rt}$.

$$y' = re^{rt}$$

$$y'' = r^2 e^{rt}$$

Plug $y, y',$ and y'' into the equation:

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$e^{rt}(ar^2 + br + c) = 0$$

$$e^{rt} \neq 0, \quad ar^2 + br + c = 0$$

Definition. An equation

$$\boxed{ar^2 + br + c = 0}$$

is called the auxiliary equation or characteristic equation associated with equation $ay'' + by' + cy = 0$.

If $b^2 - 4ac > 0$, then the auxiliary equation has two distinct real roots r_1 and r_2. Then

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is the general solution to the equation.

If $b^2 - 4ac = 0$, then the auxiliary equation has a repeated root $r \in \mathbf{R}$, $r = -\dfrac{b}{2a}$. In this case,

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} = (c_1 + c_2 t) e^{rt}$$

is the general solution to the equation.

Example 1. Find the general solution to the given equation

(a) $y'' - y' - 2y = 0$.

auxiliary equation:

$y'' \to r^2$
$y' \to r$
$y \to 1$

$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0$$
$$r_1 = 2, \quad r_2 = -1$$

General solution is: $\boxed{y(t) = c_1 e^{2t} + c_2 e^{-t}}$

(b) $y'' + 6y' + 9y = 0$.

auxiliary equation: $r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$

$r = -3$ - repeated root

General solution: $\boxed{y(t) = (c_1 + c_2 t) e^{-3t}}$

Example 2. Solve the given initial value problems.

(a) $y'' + y' = 0$, $y(0) = 2$, $y'(0) = 1$.

auxiliary equation:

$r^2 + r = 0$

$r(r+1) = 0$

$r_1 = 0, \quad r_2 = -1$

General solution: $y(t) = C_1 e^{0 \cdot t} + C_2 e^{-t}$

$= C_1 + C_2 e^{-t}$

$y'(t) = -C_2 e^{-t}$

Plug y into ICs:

$$\begin{cases} 2 = y(0) = C_1 + C_2 \\ 1 = y'(0) = -C_2 \end{cases}$$

$$\begin{cases} C_1 + C_2 = 2 \\ -C_2 = 1 \end{cases}$$

$C_2 = -1, \quad C_1 = 2 - C_2 = 3$

solution of IVP $\boxed{y(t) = 3 - e^{-t}}$

(b) $y'' - 4y' + 4y = 0$, $y(1) = 1$, $y'(1) = 1$.

auxiliary equation:

$$r^2 - 4r + 4 = 0$$
$$(r-2)^2 = 0$$
$$r = 2 \quad \text{repeated root}$$

General solution: $y(t) = (C_1 + C_2 t)e^{2t}$

$$y'(t) = C_2 e^{2t} + 2(C_1 + C_2 t)e^{2t}$$

Plug y into ICs:

$$1 = y(1) = (C_1 + C_2)e^2$$
$$1 = y'(1) = C_2 e^2 + 2(C_1 + C_2)e^2$$
$$= (2C_1 + 3C_2)e^2$$

$$\begin{cases} (C_1 + C_2)e^2 = 1 \\ (2C_1 + 3C_2)e^2 = 1 \end{cases}$$

$$\begin{cases} C_1 + C_2 = e^{-2} \\ 2C_1 + 3C_2 = e^{-2} \end{cases}$$

$$C_1 = e^{-2} - C_2$$

$$2(e^{-2} - C_2) + 3C_2 = e^{-2}$$

$$\boxed{\begin{array}{l} C_2 = -e^{-2} \\ C_1 = 2e^{-2} \end{array}}$$

Solution to IVP:

$$y(t) = (2e^{-2} - e^{-2}t)e^{2t}$$
$$= \boxed{(2-t)e^{2t-2}}$$