3.3. Complex Roots of the Characteristic Equation.

$$ay'' + by' + cy = 0$$

when $b^2 - 4ac < 0$

auxiliary equation

$$ar^2 + br + c = 0$$

has complex roots.

$i = \sqrt{-1}$ or $i^2 = -1$

any complex number

$$z = x + iy$$

where $x = Re(z)$ real part

$y = Im(z)$ imaginary part

both x and y are real

$$\sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{-1}\sqrt{4} = 2i$$

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a number $\bar{z} = x - iy$ is the complex conjugate for $z = x + iy$

$$z \cdot \bar{z} = (x - iy)(x + iy)$$
$$= x^2 - (iy)^2$$
$$= x^2 + y^2 \quad (\text{real number})$$

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

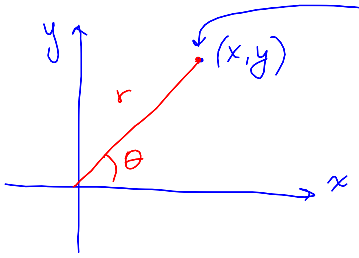
- $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

- $z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$$= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + \overbrace{i^2}^{-1} y_1 y_2$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

- $\dfrac{z_1}{z_2} = \dfrac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$

$$= \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$



$$z = x + iy$$
$$= r(\cos\theta + i\sin\theta) \quad \text{(polar form)}$$
$$r = \sqrt{x^2+y^2}$$
$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$
$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$z = x + iy$$
$$= re^{i\theta} \quad \text{(exponential form)}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler's formula:

$$\boxed{e^{a+ib} = e^{a}(\cos b + i\sin b)}$$

$$ay'' + by' + cy = 0, \quad b^2 - 4ac < 0$$

auxiliary equation

$$ar^2 + br + c = 0$$

$$r_1 = \frac{-b + \sqrt{b^2-4ac}}{2a} \qquad b^2 - 4ac = (-1)\overbrace{(4ac-b^2)}^{>0}$$

$$= -\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\, i \qquad \text{complex number}$$

$$r_2 = \frac{-b - \sqrt{b^2-4ac}}{2a} = \bar{r}_1$$

$$\mathrm{Re}(r_1) = -\frac{b}{2a} = \alpha$$

$$\mathrm{Im}(r_1) = \frac{\sqrt{4ac-b^2}}{2a} = \beta$$

Solution in the form

$$e^{r_1 t} = e^{(\alpha + i\beta)t} \overset{\text{Euler's formula}}{=} e^{\alpha t}(\cos\beta t + i \sin\beta t)$$

$\{e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t\}$ form the fundamental solution set of the equation.

General solution

$$\boxed{y(t) = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)}$$

E.1. Find the general solution of the equation

$$y'' - 2y' + 2y = 0.$$

auxiliary equation

$$r^2 - 2r + 2 = 0$$

$$r_1 = \frac{2 + \sqrt{4-8}}{2}$$

$$= \frac{2 + \sqrt{-4}}{2}$$

$$= \frac{2 + 2i}{2}$$

$$= 1 + i$$

$$Re(r_1) = 1, \quad Im(r_1) = 1$$

General solution:

$$\boxed{y(t) = (c_1 \cos t + c_2 \sin t) e^t}$$

E.2. Find the solution of the initial value problem

$$y'' + 4y' + 5y = 0, \quad y(0) = 1, \; y'(0) = 0$$

auxiliary equation

$$r^2 + 4r + 5 = 0$$

$$r_1 = \frac{-4 + \sqrt{16 - 20}}{2}$$

$$= \frac{-4 + \sqrt{-4}}{2}$$

$$= -2 + i$$

$$\operatorname{Re}(r_1) = -2, \quad \operatorname{Im}(r_1) = 1$$

General solution:

$$y(t) = (c_1 \cos t + c_2 \sin t) e^{-2t}$$

$$y'(t) = (-c_1 \sin t + c_2 \cos t) e^{-2t} + (-2)(c_1 \cos t + c_2 \sin t) e^{-2t}$$

$$1 = y(0) = c_1$$

$$0 = y'(0) = c_2 - 2c_1$$

$$\boxed{\begin{aligned} c_1 &= 1 \\ c_2 &= 2c_1 = 2 \end{aligned}}$$

Solution of IVP

$$\boxed{y(t) = (\cos t + 2 \sin t) e^{-2t}}$$