3.4 Repeated Roots; reduction of order

$$ay'' + by' + cy = 0$$

$$b^2 - 4ac = 0$$

auxiliary equation

$$ar^2 + br + c = 0$$

has a repeated root $r = -\frac{b}{2a}$

General solution of the equation

$$\boxed{y(t) = (c_1 + c_2 t)e^{-\frac{b}{2a}t}}$$

$\left\{ e^{-\frac{b}{2a}t}, \, te^{-\frac{b}{2a}t} \right\}$ fundamental solution set

Reduction of order:

Given that $y_1(t)$ is a solution of an equation

$$y'' + p(t)y' + q(t)y = 0$$

Find a second solution of the equation.

Look for a general solution in the form

$y(t) = v(t)y_1(t)$, where $v(t)$ is an unknown function

Remark: $y_1(t)$ is a solution of the equation, that is,

$$y_1'' + p(t)y_1' + q(t)y_1 = 0.$$

$$y'(t) = v'(t)y_1(t) + v(t)y_1'(t)$$

$$y''(t) = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)$$

Plug $y(t)$, $y'(t)$ and $y''(t)$ into the equation:

$$\underbrace{v''y_1 + 2v'y_1' + vy_1''}_{y''} + p(t)(\underbrace{v'y_1 + vy_1'}_{y'}) + q(t)\underbrace{v y_1}_{y} = 0$$

$$v''y_1 + v'(2y_1' + p(t)y_1) + v\underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{0} = 0$$

$$v''y_1 + v'(2y_1' + p(t)y_1) = 0$$

Substitution: $v' = w$, then $v'' = w'$

$$w'y_1 + w(2y_1' + p(t)y_1) = 0 \quad \text{(1st order linear)}$$

E.1. $ay'' + by' + cy = 0, \quad b^2 - 4ac = 0$

Given that $e^{-\frac{b}{2a}t}$ is a solution of the equation, find a second solution.

$$y(t) = v(t)e^{-\frac{b}{2a}t}$$

$$y'(t) = v'(t)e^{-\frac{b}{2a}t} - v(t)\frac{b}{2a}e^{-\frac{b}{2a}t}$$

$$y''(t) = v''(t)e^{-\frac{b}{2a}t} - \frac{2b}{2a}e^{-\frac{b}{2a}t}v'(t) + \frac{b^2}{4a^2}v(t)e^{-\frac{bt}{2a}}$$

$$a\left(v''\cancel{e^{-\frac{b}{2a}t}} - v'\frac{b}{a}\cancel{e^{-\frac{b}{2a}t}} + \frac{b^2}{4a^2}v(t)\cancel{e^{-\frac{b}{2a}t}}\right)$$

$$+ b\left(v'\cancel{e^{-\frac{b}{2a}t}} - v\frac{b}{2a}\cancel{e^{-\frac{b}{2a}t}}\right)$$

$$+ c\,v(t)\cancel{e^{-\frac{b}{2a}t}} = 0$$

$$av'' + v'\underbrace{\left(-a\frac{b}{a} + b\right)}_{0} + v\underbrace{\left(\frac{ab^2}{4a^2} - \frac{b^2}{2a} + c\right)}_{\underbrace{\frac{b^2}{4a} - \frac{b^2}{2a} + c}_{\underbrace{-\frac{b^2}{4a} + c}_{\underbrace{\frac{-b^2 + 4ac}{4a}}_{0}}}} = 0$$

$$av'' = 0$$

$$v'' = 0$$

$$v(t) = C_1 + C_2 t$$

General solution is:

$$y(t) = v(t)e^{-\frac{b}{2a}t}$$

$$= (C_1 + C_2 t)e^{-\frac{b}{2a}t}$$

$$= C_1\underbrace{e^{-\frac{b}{2a}t}}_{y_1(t)} + C_2\underbrace{t\,e^{-\frac{b}{2a}t}}_{y_2(t)}$$

E.2. Use the method of reduction of order to find a second solution of the equation

$$t^2 y'' - 4t y' + 6y = 0, \quad t > 0$$

if $y_1(t) = t^2$

$$y(t) = t^2 v(t)$$

$$y'(t) = 2t\, v(t) + t^2 v'(t)$$

$$y''(t) = 2\, v(t) + 2t\, v'(t) + 2t\, v'(t) + t^2 v''(t)$$

$$= 2v(t) + 4t\, v'(t) + t^2 v''(t)$$

$$t^2(2v + 4tv' + t^2 v'') - 4t(2tv + t^2 v') + 6t^2 v = 0$$

$$\cancel{2vt^2} + \cancel{4t^3 v'} + t^4 v'' - \cancel{8t^2 v} - \cancel{4t^3 v'} + \cancel{6t^2 v} = 0$$

$$t^4 v'' = 0$$

or $\quad v'' = 0$

$$v(t) = c_1 + c_2 t$$

General solution: $y(t) = (c_1 + c_2 t)\, t^2$

$$= c_1 \underbrace{t^2}_{y_1(t)} + c_2 \underbrace{t^3}_{y_2(t)}$$

$$\boxed{y_2(t) = t^3}$$