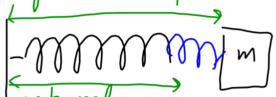
Section 3.7 **Mechanical and electrical vibrations**



A damped mass-spring oscillator consists of a mass m attached to a spring fixed at one end. Model for the motion of the mass is expressed by the initial value problem

$$\boxed{my''+by'+ky=F_{\text{external}},}\ \boxed{y(0)=y_0,}\ \boxed{y'(0)=v_0,}$$

where m is a mass, b is the damping coefficient, k is the stiffness.

Let's $F_{\text{external}}=0$

Undamped free case: $b=0$

The equation reduces to

$$my''+ky=0\,|:m$$

$$y''+\omega^2y=0$$

corresponding auxiliary eqn. is

$r^2+\omega^2=0$ or $r^2=-\omega^2$

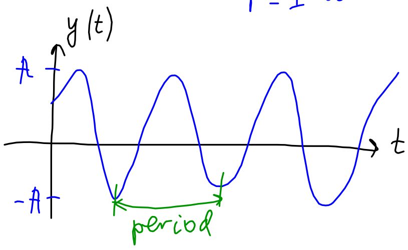
$r=\pm i\omega$

where $\boxed{\omega=\sqrt{\frac{k}{m}}}$. The solution of this equation is

$$y(t)=C_1\cos\omega t+C_2\sin\omega t$$

$$\boxed{y(t)=A\sin(\omega t+\phi),}$$

where $\boxed{A=\sqrt{C_1^2+C_2^2}}$ $\boxed{\tan\phi=\frac{C_1}{C_2}}$



The motion is periodic with

period $2\pi/\omega$

natural frequency $\omega/2\pi$

angular frequency ω

amplitude A.

Example 1. A 2-kg mass is attached to a spring with stiffness $k = 50$ N/m. The mass is displaced 1/4 m to the left of the equilibrium point and given a velocity of 1 m/sec to the left. Neglecting damping,

a) Set up an initial value problem for this system.

$$\underbrace{m}_{\text{mass}} y'' + \underbrace{\cancel{b}}_{\text{damping}}\overset{0}{} y' + \underbrace{k}_{\text{stiffness}} y = 0, \quad m = 2,\ k = 50$$

$$2y'' + 50y = 0$$

or

$$\boxed{\begin{aligned} &y'' + 25y = 0 \\ &y(0) = -0.25 \\ &y'(0) = -1 \end{aligned}}$$

b) Find the equation of motion of the mass.

$$y'' + 25y = 0$$

$$r^2 + 25 = 0, \quad r = \pm 5i$$

$$y(t) = C_1 \cos 5t + C_2 \sin 5t \ ; \quad y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t$$

$$y(0) = C_1 = -0.25$$

$$y'(0) = 5C_2 = -1 \rightarrow C_2 = -0.2$$

$$\boxed{y(t) = -0.25 \cos 5t - 0.2 \sin 5t}$$

c) Find the amplitude, period and frequency of the motion.

amplitude $A = \sqrt{C_1^2 + C_2^2}$

$$A = \sqrt{0.0625 + 0.04}$$

$$\boxed{A = \sqrt{0.1025}}$$

angular frequency $\boxed{\omega = \sqrt{\frac{k}{m}} = 5}$

natural frequency $= \dfrac{\omega}{2\pi} = \boxed{\dfrac{5}{2\pi}}$

period $= \dfrac{2\pi}{\omega} = \boxed{\dfrac{2\pi}{5}}$

d) how long does it take for the mass to pass the equilibrium point?

find t such that $y(t) = 0$

$$y(t) = -0.25 \cos 5t - 0.2 \sin 5t = 0$$

$$\frac{-0.25 \cos 5t = 0.2 \sin 5t}{0.2 \cos 5t}$$

$$\tan 5t = -\frac{0.25}{0.2} = -1.25$$

$$\boxed{t = \frac{1}{5} \tan^{-1}(1.25)}$$

Underdamped or oscillatory motion ($b^2 < 4mk$)

The solution to the equation

$$my'' + by' + ky = 0$$

auxiliary eqn. $mr^2+br+kr=0$

$$r_1 = \underbrace{\frac{-b}{2m}}_{Re(r_1)<0} + i\underbrace{\frac{\sqrt{4km-b^2}}{2m}}_{Im(r_1)}$$

is

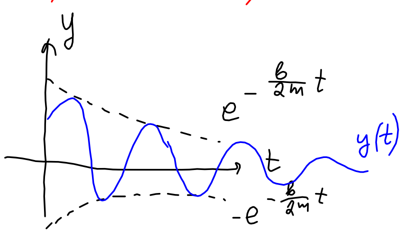
$$\boxed{y(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) = Ae^{\alpha t} \sin(\beta t + \phi),}$$

where

$\alpha = -\dfrac{b}{2m}$, $\beta = \dfrac{1}{2m}\sqrt{4mk - b^2}$, $A = \sqrt{C_1^2 + C_2^2}$, $\tan \phi = \dfrac{C_1}{C_2}$.

The solution $y(t)$ varies between $-Ae^{\alpha t}$ and $Ae^{\alpha t}$ with **quasiperiod** $\boxed{P = \dfrac{2\pi}{\beta} = \dfrac{4\pi m}{\sqrt{4mk - b^2}}}$ and **quasifrequency** $1/P$. $y(t) \to 0$ as $t \to \infty$.

An exponential factor $Ae^{\alpha t}$ is called a **damping factor**.



The system is called **underdamped** because there is not enough damping present (b is too small) to prevent the system from oscillating.

Overdamped motion ($b^2 > 4mk$) no oscillation

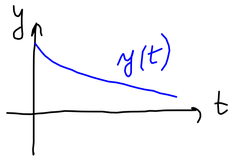
The solution to the equation

$$my'' + by' + ky = 0$$

the auxiliary eqn has two distinct real roots.

is

$$\boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},}$$



where $r_1 = -\dfrac{b}{2m} + \dfrac{1}{2m}\sqrt{4mk - b^2}$, $r_2 = -\dfrac{b}{2m} - \dfrac{1}{2m}\sqrt{4mk - b^2}$

$r_2 < 0$, and since $b^2 > b^2 - 4mk$, $r_1 < 0$. $y(t) \to 0$ as $t \to \infty$.

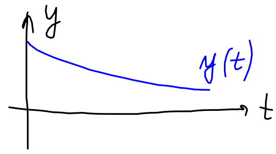
Critically damped motion ($b^2 = 4mk$) The solution to the equation

$$my'' + by' + ky = 0$$

auxiliary eqn has a repeated root $r = -\dfrac{b}{2m} < 0$

is

$$\boxed{y(t) = (c_1 + c_2 t)e^{-\frac{b}{2m}t}.}$$



$y(t) \to 0$ as $t \to \infty$. no oscillation

Example 2. The motion of the mass-spring system with damping is governed by

$$y''(t) + by'(t) + 64y(t) = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Find the equation of the motion for $b = 10, 16, 20$.

$b^2 - 4mk$

$b^2 - 4(64)$

$b^2 - 256$

$b=10$ (underdamped motion) $(b^2 - 256 < 0)$

$b=16$ (critically damped motion) $(b^2 - 256 = 0)$

$b=20$ (overdamped motion) $(b^2 - 256 > 0)$

$b=10$

auxiliary eqn: $r^2 + 10r + 64 = 0$

roots: $r_1 = \dfrac{-10 + \sqrt{100-256}}{2}$

$$= \underbrace{-5}_{Re(r_1)} + i \underbrace{\frac{\sqrt{156}}{2}}_{Im(r_1)}$$

General solution: $y(t) = e^{-5t}\left(c_1 \cos \dfrac{\sqrt{156}}{2}t + c_2 \sin \dfrac{\sqrt{156}}{2}t\right)$

$y(0) = \boxed{c_1 = 1}$

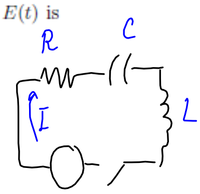
$y'(0) = -5c_1 + c_2 \dfrac{\sqrt{156}}{2} = 0$

$$\boxed{c_2 = +\frac{10}{\sqrt{156}}}$$

$$\boxed{y(t) = e^{-5t}\left(\cos \frac{\sqrt{156}}{2}t + \frac{10}{\sqrt{156}} \sin \frac{\sqrt{156}}{2}t\right)}$$

Electric circuits.

If Q is the charge at time t in an electrical closed circuit with inductance L, resistance R, and capacitance C, then by Kirchhoff's Second Law (from Physics) the impressed voltage $E(t)$ is equal to the sum of the voltage drops in the rest of the circuit



$$E(t) = IR + \frac{Q}{C} + LI'(t)$$

By substitution $I = Q'$ we get

$$\boxed{LQ'' + RQ' + \frac{1}{C}Q = E(t)}$$

Analogy between electrical and mechanical quantities:

Charge Q	Position $y(t)$
Inductance L	mass m
Resistance R	Damping constant b
Inverse capacitance $1/C$	~~Spring constant k~~ stiffness k
Impressed voltage $E(t)$ (electromotive force)	External force $F(t)$