Section 3.8 **Forced vibrations**

Let's investigate the effect of a cosine forcing function on the system governed by the differential equation

$$my'' + by' + ky = F_0 \cos \gamma t,$$

where F_0, γ are nonnegative constants and $b^2 < 4mk$ (the system is underdamped).

The general solution to this equation is

$$y(t) = Ae^{-(b/2m)t} \sin\left(\frac{\sqrt{4mk - b^2}}{2m} t + \phi\right) +$$

$$+\frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} \sin(\gamma t + \theta),$$

where A, ϕ are constants and $\tan\theta = \frac{k - m\gamma^2}{b\gamma}$.

$$y_h(t) = Ae^{-(b/2m)t} \sin\left(\frac{\sqrt{4mk - b^2}}{2m} t + \phi\right)$$

is called the **transient part** of solution. $y_h \to 0$ as $t \to \infty$.

$$y_p(t) = \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} \sin(\gamma t + \theta)$$

is the offspring of the external forcing function $f(t) = F_0 \cos \gamma t$. y_p is sinusoidal with angular frequency γ. y_p is out of phase with $f(t)$ by the angle $\theta - \pi/2$, and its magnitude is different by the factor

$$\frac{1}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}$$

y_p is called the **steady-state** solution.

The factor $\dfrac{1}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}$ is called the **frequency gain** or **gain factor**.

Example 1. A 2-kg mass is attached to a spring with stiffness $k = 45$ N/m. At time $t = 0$, an external force $f(t) = 12\cos 3t$ is applied to the system. The damping constant for the system is 4 N-sec/m. Determine the steady-state solution for the system.

In general, the amplitude of the steady-state solution depends on γ and is given by

$$A(\gamma) = F_0 M(\gamma) = \frac{1}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}$$

$M(\gamma)$ is the frequency gain. The graph of $M(\gamma)$ is called the **frequency response curve** or **resonance curve** for the system.

$M(0) = \frac{1}{k}$, $M(\gamma) \to 0$ as $\gamma \to \infty$.

$$M'(\gamma) = -\frac{\gamma(b^2 - 2m(k - m\gamma^2))}{[(k - m\gamma^2)^2 + b^2\gamma^2]^{3/2}} = 0$$

if and only if

$$\gamma = 0 \quad \text{or} \quad \gamma = \gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$$

When the system is critically damped or overdamped, $M'(\gamma) = 0$ only when $\gamma = 0$. In this case $M(\gamma)$ increases from $1/k$ to 0 as $\gamma \to \infty$.

When $b^2 - 2mk < 0$, then $M'(\gamma) = 0$ at $\gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$, then

$$M(\gamma_r) = \frac{1}{b\sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}}$$

The value $\gamma_r/2\pi$ is called the **resonance frequency** for the system. When the system is stimulated by an external force as this frequency, it is said to be **at resonance**.

Let $k = m = 1$, then $M(\gamma) = \frac{1}{\sqrt{(1-\gamma^2)^2 + b^2\gamma^2}}$, for $b < \sqrt{2}$ the resonance frequency is

$$\gamma_r = \frac{1}{2\pi}\sqrt{1 - \frac{b^2}{2}}$$

As $b \to 0$ $\gamma_r/2\pi \to \sqrt{k}/m/2\pi = 1/2\pi$. $1/2\pi$ is the natural frequency for the undamped system.

Consider the undamped system ($b = 0$) with forcing term $F_0 \cos\gamma t$. This system is governed by

$$m\frac{d^2y}{dt^2} + ky = F_0 \cos\gamma t$$

$$y(t) = y_h(t) + y_p(t)$$

$$y_h(t) = A\sin(\omega t + \phi), \quad \omega = \sqrt{k/m}$$

If $\gamma \neq \omega$

$$y_p(t) = \frac{F_0}{k - m\gamma^2}\sin(\gamma t + \theta)$$

If $\gamma = \omega$

$$y_p(t) = \frac{F_0}{2m\omega} t \sin\omega t$$

Hence, in the *undamped resonant case* ($\gamma = \omega$),

$$y(t) = A\sin(\omega t + \phi) + \frac{F_0}{2m\omega} t \sin \omega t$$

$y_p(t)$ oscillates between $-\dfrac{F_0}{2m\omega}$ and $\dfrac{F_0}{2m\omega}$. As $t \to \infty$, the maximum magnitude of $y_p(t) \to \infty$.

If the damping constant b is very small, the system is subject to large oscillations when the forcing function has a frequency near the resonance frequency for the system.

When the mass-spring system is **hung vertically**, the gravitational force can be ignored if $y(t)$ is measured from the equilibrium position. The equation of motion for this system is

$$my'' + by' + ky = F_{\text{external}},$$

where m is a mass, b is the damping coefficient, k is the stiffness.

Example 2. A 2-kg mass is attached to a spring hanging from the ceiling, hereby causing the spring to stretch 20 cm upon coming to rest at equilibrium. At time $t = 0$ the mass is displaced 5 cm below the equilibrium position and released. At this same instant, an external force $f(t) = 0.3\cos t$ N is applied to the system. If the damping constant to the system is 5 N-sec/m, determine the equation of motion for the mass. What is the resonance frequency for the system?