Section 6.4 **Differential equations with discontinuous forcing functions.**

Important formulas:

1. $\mathcal{L}\{y'\}(s) = s\mathcal{L}\{y\}(s) - y(0)$
2. $\mathcal{L}\{y''\}(s) = s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0)$
3. $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$
4. $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$
5. $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$, where $F(s) = \mathcal{L}\{f(t)\}$

**Example 1.** Solve the initial value problem.

$$y'' + 5y' + 6y = g(t), \quad y(0) = 0, y'(0) = 2,$$

where $g(t) = \begin{cases} 0, & 0 \le t < 1, \\ t, & 1 < t < 5, \\ 1, & 5 < t. \end{cases}$

$\mathcal{L}\{y'' + 5y' + 6y\} = \mathcal{L}\{g(t)\}$

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y'\} = sY(s) - y(0)$

$= sY(s)$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$

$= s^2Y(s) - 2$

$\mathcal{L}\{g(t)\} - ?$

1) Express $g(t)$ in terms of $u_c(t)$

$g(t) = 0 + u_1(t)(t-0) + u_5(t)(1-t)$

$= tu_1(t) + (1-t)u_5(t)$

2) $\mathcal{L}\{g(t)\}$

$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$, $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$

need $(t-1)$ need $(t-5)$

$= \mathcal{L}\{t\,u_1(t) + (1-t)u_5(t)\}$

$= \mathcal{L}\{((t-1)+1)u_1(t) - ((t-5)+4)u_5(t)\}$

$= \mathcal{L}\{\underbrace{(t-1)}_{f(t-1)}u_1(t)\} + \mathcal{L}\{u_1(t)\} - \mathcal{L}\{\underbrace{(t-5)}_{f(t-5)}u_5(t)\} - 4\mathcal{L}\{u_5(t)\}$

$f(t-1) = t-1 \Rightarrow f(t) = t$, $\mathcal{L}\{t\} = \frac{1}{s^2}$

$\mathcal{L}\{(t-1)u_1(t)\} = e^{-s}\mathcal{L}\{t\} = \dfrac{e^{-s}}{s^2}$

$f(t-5) = t-5 \rightarrow f(t) = t$, $\mathcal{L}\{t\} = \frac{1}{s^2}$

$\mathcal{L}\{(t-5)u_5(t)\} = e^{-5s}\mathcal{L}\{t\} = \dfrac{e^{-5s}}{s^2}$

$\mathcal{L}\{g(t)\} = \dfrac{e^{-s}}{s^2} + \dfrac{e^{-s}}{s} - \dfrac{e^{-5s}}{s^2} - 4\dfrac{e^{-5s}}{s}$

$= e^{-s}\cdot\dfrac{1+s}{s^2} - e^{-5s}\,\dfrac{4s+1}{s^2}$

$$s^2Y(s)-2+5sY(s)+6Y(s)=e^{-s}\frac{1+s}{s^2}-e^{-5s}\frac{4s+1}{s^2}$$

$$Y(s)(s^2+5s+6)=2+e^{-s}\frac{1+s}{s^2}-e^{-5s}\frac{4s+1}{s^2}$$

$$Y(s)=\frac{2}{s^2+5s+6}+e^{-s}\frac{1+s}{s^2(s^2+5s+6)}-e^{-5s}\frac{4s+1}{s^2(s^2+5s+6)}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s^2+5s+6}+e^{-s}\frac{1+s}{s^2(s^2+5s+6)}-e^{-5s}\frac{4s+1}{s^2(s^2+5s+6)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{2}{s^2+5s+6}\right\}+\mathcal{L}^{-1}\left\{e^{-s}\frac{1+s}{s^2(s^2+5s+6)}\right\}-\mathcal{L}\left\{e^{-5s}\frac{4s+1}{s^2(s^2+5s+6)}\right\}$$

$$\frac{2}{s^2+5s+6}=\frac{2}{(s+2)(s+3)}=2\left[\frac{1}{s+2}-\frac{1}{s+3}\right]$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{2}{s^2+5s+6}\right\}=2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}=2e^{-2t}-2e^{-3t}}$$

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{1+s}{s^2(s^2+5s+6)}\right\}$$

Find $\mathcal{L}^{-1}\left\{\frac{1+s}{s^2(s^2+5s+6)}\right\}$

Partial fractions: $\frac{1+s}{s^2(s^2+5s+6)}=\frac{1+s}{s^2(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}+\frac{D}{s+3}$

$$\frac{1+s}{s^2(s+2)(s+3)}=\frac{As(s+2)(s+3)+B(s+2)(s+3)+Cs^2(s+3)+Ds^2(s+2)}{s^2(s+2)(s+3)}$$

$s=0$: $1=6B$ $\quad\boxed{B=1/6}$

$s=-2$: $-1=4C$ $\quad\boxed{C=-1/4}$

$s=-3$: $-2=-9D$ $\quad\boxed{D=2/9}$

$s=-1$: $0=-2A+2B+2C+D$

$$2A=2B+2C+D=\frac{1}{3}-\frac{1}{2}+\frac{2}{9}=\frac{1}{18}$$

$$\boxed{A=\frac{1}{36}}$$

$$\mathcal{L}^{-1}\left\{\frac{1+s}{s^2(s^2+5s+6)}\right\}=\frac{1}{36}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{1}+\frac{1}{6}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}}_{t}-\frac{1}{4}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}}_{e^{-2t}}+\frac{2}{9}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}}_{e^{-3t}}$$

$$=\frac{1}{36}+\frac{1}{6}t-\frac{1}{4}e^{-2t}+\frac{2}{9}e^{-3t}$$

$\mathcal{L}^{-1}\{e^{-cs}F(s)\}=u_c(t)f(t-c)$, where $F(s)=\mathcal{L}\{f(t)\}$

$$\boxed{\mathcal{L}^{-1}\left\{e^{-s}\frac{1+s}{s^2(s^2+5s+6)}\right\}=u_1(t)\left[\frac{1}{36}+\frac{t-1}{6}-\frac{e^{-2(t-1)}}{4}+\frac{2e^{-3(t-1)}}{9}\right]}$$

$$\mathcal{L}^{-1}\left\{e^{-5s}\frac{4s+1}{s^2(s^2+5s+6)}\right\}$$

Partial fractions: $\frac{4s+1}{s^2(s^2+5s+6)}=\frac{E}{s}+\frac{F}{s^2}+\frac{H}{s+2}+\frac{G}{s+3}$

$$\mathcal{L}^{-1}\left\{\frac{4s+1}{s^2(s^2+5s+6)}\right\}=E+Ft+He^{-2t}+Ge^{-3t}$$

$$\boxed{\mathcal{L}^{-1}\left\{e^{-5s}\frac{4s+1}{s^2(s^2+5s+6)}\right\}=u_5(t)\left[E+F(t-5)+He^{-2(t-5)}+Ge^{-3(t-5)}\right]}$$

Section 6.5 **Impulse Functions.**

**Definition.** A unit impulse function (Dirac delta function) is a function defined by

$$\boxed{\delta(t)=\begin{cases}1, & t=0,\\ 0, & t\neq 0.\end{cases}}$$

$$\delta(t-t_0)=\begin{cases}1, & t=t_0\\ 0, & t\neq t_0\end{cases}$$

**Property of the unit impulse function:**

$$\boxed{\int_{-\infty}^{\infty}\delta(t)dt=1.}$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\}=e^{-st_0}}$$

$$\mathcal{L}^{-1}\{e^{-st_0}\}=\delta(t-t_0)$$

**Example 1.** Solve the initial value problem.

$$y''+2y'+2y=\delta(t-\pi),\qquad y(0)=1, y'(0)=0.$$

$$\mathcal{L}\{y''+2y'+2y\}=\mathcal{L}\{\delta(t-\pi)\}$$

$$\mathcal{L}\{y\}=Y(s)$$

$$\mathcal{L}\{y'\}=sY(s)-y(0)$$
$$=sY(s)-1$$

$$\mathcal{L}\{y''\}=s^2Y(s)-sy(0)-y'(0)$$
$$=s^2Y(s)-s$$

$$\mathcal{L}\{\delta(t-\pi)\}=e^{-s\pi}$$

$$s^2Y(s)-s+2sY(s)-2+2Y(s)=e^{-\pi s}$$

$$Y(s)(s^2+2s+2)=s+2+e^{-\pi s}$$

$$Y(s)=\frac{s+2}{s^2+2s+2}+\frac{e^{-\pi s}}{s^2+2s+2}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+2s+2}\right\}+\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+2s+2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+2s+2}\right\}=\mathcal{L}^{-1}\left\{\frac{\overbrace{s+2}^{(s+1)+1}}{(s+1)^2+1}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$=e^{-t}\cos t+e^{-t}\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}=e^{-t}\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+2s+2}\right\}=\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{(s+1)^2+1}\right\}$$

$$=u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)$$

$$\boxed{y(t)=u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)+e^{-t}(\cos t+\sin t)}$$