Gauss Elimination Method

The Gauss Method is a suitable technique for solving systems of linear equations of any size. A sequence of operations (see below) of the Gauss-Jordan elimination method allows us to obtain at each step an equivalent system - that is, a system having the same solution as the original system.

The operations of the Gauss-Jordan elimination method are

1. Interchange any two equations.
2. Replace an equation by a nonzero multiple of itself.
3. Replace an equation by itself plus a nonzero multiple of any other equation.

An augmented matrix that is formed by combining the coefficient matrix and the constant matrix. For example, for the system of linear equations $\begin{cases} 3x_1 + 12x_2 = 20 \\ 2x_2 = x_1 + 7 \end{cases}$ the augmented matrix is $\left(\begin{array}{cc|c} 3 & 12 & 20 \\ -1 & +2 & 7 \end{array}\right)$.

The goal of the Gauss Elimination Method is to get the augmented matrix into **Reduced Echelon Form**.

A matrix is in row echelon form if

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (all zero rows, if any, belong at the bottom of the matrix).
- The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
- All entries in a column below a leading entry are zeroes (implied by the first two criteria).

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

augmented matrix $\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & & & & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{array}\right)$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & & & & \ldots \\ a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n} & b_{n-1} \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{array}\right) \xrightarrow{\text{Gauss Elimination method}} \left(\begin{array}{ccccc|c} c_{11} & c_{12} & \ldots & c_{1,n-1} & c_{1n} & d_1 \\ 0 & c_{22} & \ldots & c_{2,n-1} & c_{2n} & d_2 \\ \ldots & & & & & \ldots \\ 0 & 0 & \ldots & c_{n-1,n-1} & c_{n-1,n} & d_{n-1} \\ 0 & 0 & \ldots & 0 & c_{nn} & d_n \end{array}\right)$$

row echelon form

zeroes below the main diagonal

To put a matrix in Reduced Form, there are three valid Row Operations:

1. Interchange any two rows ($R_i \leftrightarrow R_j$).
2. Replace any row by a nonzero constant multiple of itself ($R_i \leftrightarrow cR_i$).
3. Replace any row by the sum of that row and a constant multiple of any other row $R_i \leftrightarrow (R_i + cR_j)$.

Eigenvalues and Eigenvectors

Definition. A number λ is called an eigenvalue of matrix A if there exists a **nonzero** vector v such that

$$Av = \lambda v,$$

and v is called an eigenvector corresponding to the eigenvalue λ. **Example.** If A is diagonal matrix,

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

then the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues and the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad v_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

are the corresponding eigenvectors.

Eigenvalues are solutions of the following **characteristic equation** (polynomial):

$$\det(A - \lambda I) = 0.$$

The characteristic equation in the case $n = 2$ can be found as

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A) = 0.$$

Remark. For $n \times n$ matrix the characteristic equation is a polynomial equation of degree n. The eigenvectors corresponding to λ can be found by solving the corresponding system of linear equations $(A - \lambda I)v = 0$ (as we will see in the next.

Example 1. Find eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$.

$$A - \lambda I = \begin{pmatrix} -2-\lambda & 1 \\ 2 & -3-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (-2-\lambda)(-3-\lambda) - 2$$
$$= 6 + 5\lambda + \lambda^2 - 2$$
$$= \lambda^2 + 5\lambda + 4 = 0$$

solve for λ

$\lambda_1 = -1,\ \lambda_2 = -4$ eigenvalues

Find eigenvectors.

$\lambda_1 = -1$ corresponding eigenvector is $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\vec{v}$ is a solution of the system

$$(A - \underbrace{\lambda_1}_{-1} I)\vec{v} = \vec{0}$$

$$(A + I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} -2+1 & 1 \\ 2 & -3+1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

component form: $\begin{cases} -v_1 + v_2 = 0 \quad (-1) \\ 2v_1 - 2v_2 = 0 \quad (\div 2) \end{cases}$

$\begin{cases} v_1 - v_2 = 0 \\ v_1 - v_2 = 0 \end{cases}$ underdetermined system has infinitely many solutions

$v_1 = v_2$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \underset{v_1 = 1}{\overset{\text{plug}}{=}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to $\lambda = -1$

$\lambda_2 = -4$. eigenvector $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$\vec{w}$ is the solution of the system

$$(A - \underbrace{\lambda_2}_{-4} I)\vec{w} = \vec{0}$$

$$(A + 4I)\vec{w} = \vec{0}$$

$$\begin{pmatrix} -2+4 & 1 \\ 2 & -3+4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

component form: $\begin{cases} 2w_1 + w_2 = 0 \\ 2w_1 + w_2 = 0 \end{cases}$ underdetermined system

$w_2 = -2w_1$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ -2w_1 \end{pmatrix} = w_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \underset{}{\overset{\text{plug } w_1 = 1}{=}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ corresponds to $\lambda_2 = -4$

Example 2. Given

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

1. Find eigenvalues of A.

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-1-\lambda) + 2 + 12 - 8(2-\lambda) - 3(1+\lambda) + (1-\lambda) = 0$$

$$-(1-\lambda)(1+\lambda)(2-\lambda) + 14 - 16 + 8\lambda - 3 - 3\lambda + 1 - \lambda = 0$$

$$-(1-\lambda^2)(2-\lambda) - 4 + 4\lambda = 0$$

$$-(2 - \lambda - 2\lambda^2 + \lambda^3) - 4 + 4\lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

~~2. Find eigenvectors of A~~

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$\lambda = 1$: $\quad 1 - 2 - 5 + 6 \neq 0$

$$\begin{array}{r|l} & \lambda^2 - \lambda - 6 \\ \lambda - 1 & \lambda^3 - 2\lambda^2 - 5\lambda + 6 \\ & \underline{\lambda^3 - \lambda^2} \\ & -\lambda^2 - 5\lambda \\ & \underline{-\lambda^2 + \lambda} \\ & -6\lambda + 6 \end{array}$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = (\lambda - 1)(\lambda^2 - \lambda - 6)$$

$$= (\lambda - 1)(\lambda - 3)(\lambda + 2) = 0$$

$\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = -2$ eigenvalues.