Section 7.5 **Homogeneous linear systems with constant coefficients**
Section 7.6 **Complex eigenvalues**
Section 7.8 **Repeated eigenvalues**

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

here $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$.

- **Real Distinct Eigenvalues.** If $A$ has $n$ **distinct real** eigenvalues $\lambda_1, \ldots, \lambda_n$ and $v_1, \ldots, v_n$ are the corresponding eigenvectors, then

$$\{e^{\lambda_1 t} v_1, \ldots, e^{\lambda_n t} v_n\}$$

is the fundamental solution set and the general solution is

$$X(t) = C_1 e^{\lambda_1 t} v_1 + \ldots + C_n e^{\lambda_n t} v_n.$$

Example 1. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$$

- **Complex eigenvalues.** Any complex eigenvalue must occur in conjugate pairs. If $\lambda = \alpha + i\beta$ is an eigenvalue of the matrix $\mathbf{A}$, then so is $\bar{\lambda} = \alpha - i\beta$.

  If $\vec{v} = \vec{u} + i\vec{w}$ is an eigenvector corresponding to $\lambda = \alpha + i\beta$, then $\bar{\vec{v}} = \vec{u} - i\vec{w}$ is an eigenvector corresponding to $\bar{\lambda} = \alpha - i\beta$. Then the corresponding solution of the system is

$$\begin{aligned}\vec{x}(t) &= (\vec{u} + i\vec{w})e^{(\alpha+i\beta)t} = (\vec{u} + i\vec{w})(\cos\beta t + i\sin\beta t)e^{\alpha t} \\ &= (\vec{u}\cos\beta t - \vec{w}\sin\beta t))e^{\alpha t} + i\,(\vec{u}\sin\beta t + \vec{w}\cos\beta t))e^{\alpha t}\end{aligned}$$

  Then the vectors

$$\begin{aligned}\vec{x}_1(t) &= Re(\vec{x}(t)) = (\vec{u}\cos\beta t - \vec{w}\sin\beta t))e^{\alpha t} \\ \vec{x}_2(t) &= Im(\vec{x}(t)) = (\vec{u}\sin\beta t + \vec{w}\cos\beta t))e^{\alpha t}\end{aligned}$$

  are real valued solutions of the system.

  **Example 2.** Find the general solution of the system

  1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

- **Repeated eigenvalues.** Let $\lambda_i$ be an eigenvalue of the matrix $\mathbf{A}$ of multiplicity $1 < k \leq n$. Then there are $k$ linearly independent eigenvectors $\vec{v}_1(t)$,...,$\vec{v}_1(t)$ corresponding to $\lambda$, if $k < n$. If $k = n$, then there is only one vector $\vec{v_1}(t)$ corresponding to $\lambda$. The remaining $n-1$ vectors corresponding to $\lambda$ are solutions to the system

$$(\mathbf{A} - \lambda\mathbf{I})\vec{v}_{i+1}(t) = \vec{v_i}(t), \quad i = 1, 2, ..., n-1$$

Vectors $\vec{v}_2(t)$, ... $\vec{v}_{n-1}(t)$ are called **generalized eigenvectors** corresponding to $\lambda$.

Then the corresponding solutions of the system are $v_1(t)e^{\lambda t}$, $tv_2(t)e^{\lambda t}$,..., $t^k v_k(t)e^{\lambda t}$.

**Example 3.** Find the general solution of the system.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}$