

Section 7.5 Homogeneous linear systems with constant coefficients
 Section 7.6 Complex eigenvalues
 Section 7.8 Repeated eigenvalues

$$x' = Ax,$$

here $x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$.

- **Real Distinct Eigenvalues.** If A has n distinct real eigenvalues $\lambda_1, \dots, \lambda_n$ and v_1, \dots, v_n are the corresponding eigenvectors, then

$$\{e^{\lambda_1 t} v_1, \dots, e^{\lambda_n t} v_n\}$$

is the fundamental solution set and the general solution is

$$X(t) = C_1 e^{\lambda_1 t} v_1 + \dots + C_n e^{\lambda_n t} v_n.$$

Example 1. Find the general solution of the system

$$x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x, \quad A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}$$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 4 = 0$$

$$-6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = -1 \quad \text{eigen values.}$$

Corresponding eigenvectors:

1) $\lambda_1 = 2$ $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{v}$ is a solution of

$$(A - 2I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{cases} v_1 - 2v_2 = 0 \\ 2v_1 - 4v_2 = 0 \end{cases}$$

$$v_1 = 2v_2$$

$$\vec{v} = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} \stackrel{v_2=1}{=} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

2) $\lambda_2 = -1$ $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \vec{w}$ is a solution of

$$(A + I)\vec{w} = \vec{0}$$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{cases} 4w_1 - 2w_2 = 0 \\ 2w_1 - w_2 = 0 \end{cases}$$

$$w_2 = 2w_1$$

$$\vec{w} = \begin{pmatrix} w_1 \\ 2w_1 \end{pmatrix} \stackrel{w_1=1}{=} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

General solution: $\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$

• **Complex eigenvalues.** Any complex eigenvalue must occur in conjugate pairs. If $\lambda = \alpha + i\beta$ is an eigenvalue of the matrix A , then so is $\bar{\lambda} = \alpha - i\beta$.

If $\vec{v} = \vec{u} + i\vec{w}$ is an eigenvector corresponding to $\lambda = \alpha + i\beta$, then $\vec{v} = \vec{u} - i\vec{w}$ is an eigenvector corresponding to $\bar{\lambda} = \alpha - i\beta$. Then the corresponding solution of the system is

$$\begin{aligned}\vec{x}(t) &= (\vec{u} + i\vec{w})e^{(\alpha+i\beta)t} = (\vec{u} + i\vec{w})(\cos \beta t + i \sin \beta t)e^{\alpha t} \\ &= (\vec{u} \cos \beta t - \vec{w} \sin \beta t)e^{\alpha t} + i(\vec{u} \sin \beta t + \vec{w} \cos \beta t)e^{\alpha t}\end{aligned}$$

Then the vectors

$$\begin{aligned}\vec{x}_1(t) &= \text{Re}(\vec{x}(t)) = (\vec{u} \cos \beta t - \vec{w} \sin \beta t)e^{\alpha t} \\ \vec{x}_2(t) &= \text{Im}(\vec{x}(t)) = (\vec{u} \sin \beta t + \vec{w} \cos \beta t)e^{\alpha t}\end{aligned}$$

are real valued solutions of the system.

Example 2. Find the general solution of the system

$$1. \quad \vec{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x} \quad A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix}$$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{aligned}\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} &= 0 \\ (3-\lambda)(-1-\lambda) + 8 &= 0 \\ -3 - 3\lambda + \lambda^2 + 8 &= 0 \\ \lambda^2 - 3\lambda + 5 &= 0\end{aligned}$$

$$\lambda_1 = \frac{3 + \sqrt{4-20}}{2} = \frac{3 + \sqrt{-16}}{2} = \frac{3 + 4i}{2} = 1.5 + 2i$$

$$\lambda_2 = 1 - 2i$$

Corresponding eigenvectors: $(A - \lambda_1 I)\vec{v} = \vec{0}$ is a solution of $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{aligned}\begin{pmatrix} 3-(1+2i) & -2 \\ 4 & -1-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}\end{aligned}$$

Component form: $\begin{cases} (2-2i)v_1 - 2v_2 = 0 \\ 4v_1 - (2+2i)v_2 = 0 \end{cases}$

$$v_2 = (-i)v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ (1-i)v_1 \end{pmatrix} \stackrel{v_1=1}{=} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{v} e^{\lambda_1 t} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] e^{(1+2i)t} \quad e^{(1+2i)t} = e^t (\cos 2t + i \sin 2t)$$

$$= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] e^t (\cos 2t + i \sin 2t)$$

$$= e^t \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} i \sin 2t + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t + i^2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t \right]$$

$$= e^t \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 2t + i \left\{ \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t \right\} \right]$$

$$= e^t \left[\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix} \right]$$

$$\vec{x}_1 = \text{Re}(\vec{v} e^{\lambda_1 t}) = e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}$$

$$\vec{x}_2 = \text{Im}(\vec{v} e^{\lambda_1 t}) = e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$$

General solution: $\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 = c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}$

$$\vec{x}' = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix} \vec{x}$$

eigenvalues: $\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$

corresponding eigenvector $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$\vec{v} e^{\lambda_1 t} = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right] e^{(-1+2i)t}$$

$$e^{(-1+2i)t} = e^{-t} (\cos 2t + i \sin 2t)$$

$$= \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right] e^{-t} (\cos 2t + i \sin 2t)$$

$$= e^{-t} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t + i \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t + i^2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right]$$

$$= e^{-t} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t + i \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right] \right\}$$

$$= e^{-t} \left(\begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix} \right)$$

$$\vec{x}_1(t) = \operatorname{Re}(\vec{v} e^{\lambda_1 t}) = e^{-t} \begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\vec{x}_2(t) = \operatorname{Im}(\vec{v} e^{\lambda_1 t}) = e^{-t} \begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix}$$

$$\text{General solution: } \vec{x}(t) = C_1 e^{-t} \begin{pmatrix} -2 \sin 2t \\ \cos 2t \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 2 \cos 2t \\ \sin 2t \end{pmatrix}$$

- **Repeated eigenvalues.** Let λ_i be an eigenvalue of the matrix \mathbf{A} of multiplicity $1 < k \leq n$. Then there are k linearly independent eigenvectors $\vec{v}_1(t), \dots, \vec{v}_k(t)$ corresponding to λ , if $k < n$. If $k = n$, then there is only one vector $\vec{v}_1(t)$ corresponding to λ . The remaining $n - 1$ vectors corresponding to λ are solutions to the system

$$(\mathbf{A} - \lambda \mathbf{I})\vec{v}_{i+1}(t) = \vec{v}_i(t), \quad i = 1, 2, \dots, n - 1$$

Vectors $\vec{v}_2(t), \dots, \vec{v}_{n-1}(t)$ are called **generalized eigenvectors** corresponding to λ .

Then the corresponding solutions of the system are $v_1(t)e^{\lambda t}, tv_2(t)e^{\lambda t}, \dots, t^k v_k(t)e^{\lambda t}$.

Example 3. Find the general solution of the system.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

$$2. x' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} x \quad A = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 \\ -4 & 1-\lambda & 0 \\ 3 & 6 & 2-\lambda \end{pmatrix}$$

Eigenvalues. $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -4 & 1-\lambda & 0 \\ 3 & 6 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) = 0$

$\lambda_1 = 1$ - repeated of multiplicity 2
 $\lambda_2 = 2$ - nonrepeated.

corresponding eigenvectors.

1) $\lambda_2 = 2$ corresponding eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is a solution of $(A - \lambda_2 I) \vec{v} = \vec{0}$

$$\begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \begin{cases} -v_1 = 0 \\ -4v_1 - v_2 = 0 \\ 3v_1 + 6v_2 = 0 \end{cases}$$

$v_1 = v_2 = 0, v_3$ can be any number

$$\vec{v} = \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \stackrel{v_3=1}{=} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = 2$$

2) $\lambda_1 = 1$ - repeated, of multiplicity 2.

$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ is a solution of $(A - \lambda_1 I) \vec{y} = \vec{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \begin{cases} -4y_1 = 0 \\ 3y_1 + 6y_2 + y_3 = 0 \end{cases}$$

$y_1 = 0, y_3 = -6y_2$

$$\vec{y} = \begin{pmatrix} 0 \\ y_2 \\ -6y_2 \end{pmatrix} \stackrel{y_2=1}{=} \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$$

2nd eigenvector corresponding to $\lambda_1 = 1$ $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

is a solution of the system

$$(A - I) \vec{w} = \vec{y}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}; \quad \begin{cases} -4w_1 = 1 \\ 3w_1 + 6w_2 + w_3 = -6 \end{cases}$$

$$w_1 = -1/4$$

$$w_3 = -6 - 3w_1 - 6w_2 = -6 + \frac{3}{4} - 6w_2 = -\frac{21}{4} - 6w_2$$

$$w_3 = -\frac{21}{4} - 6w_2$$

$$\vec{w} = \begin{pmatrix} -1/4 \\ w_2 \\ -\frac{21}{4} - 6w_2 \end{pmatrix} \stackrel{w_2=0}{=} \begin{pmatrix} -1/4 \\ 0 \\ -\frac{21}{4} \end{pmatrix}$$

general solution: $\vec{x}(t) = C_1 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} + C_2 e^t \begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}$