Section 3.2 Solutions of linear homogeneous equations; the Wronskian.

A second order ordinary differential equation has the form
\[ \frac{d^2 y}{dt^2} = f(t, y, \frac{dy}{dt}) \]
where \( f \) is some given function.

An initial value problem consists of a differential equation together with the pair of initial conditions
\[ y(t_0) = y_0, \quad y'(t_0) = y_1. \]

A second order ordinary differential equation is said to be linear if it is written in the form
\[ P(t)y'' + Q(t)y' + R(t)y = G(t) \]
or
\[ \frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = g(t). \tag{1} \]

If \( g(t) = 0 \), then the equation
\[ y'' + p(t)y' + q(t)y = 0 \tag{2} \]
is called homogeneous. Otherwise, the equation is called nonhomogeneous.

**Theorem 1 (existence and uniqueness of solution).** Suppose \( p(t), q(t), \) and \( g(t) \) are continuous on some interval \((a, b)\) that contains the point \( t_0 \). Then, for any choice of initial values \( y_0, y_1 \) there exists a unique solution \( y(t) \) on the whole interval \((a, b)\) to the initial value problem
\[ y'' + p(t)y' + q(t)y = g(t), \]
\[ y(t_0) = y_0, \quad y'(0) = y_1. \]

**Example 1.** Find the largest interval for which Theorem 2 ensures the existence and uniqueness of solution to the initial value problem
\[ e^t y'' - \frac{y'}{t-3} + y = \ln t, \]
\[ y(1) = y_0, \quad y'(1) = y_1, \]
where \( y_0 \) and \( y_1 \) are real constants.

**Theorem 2 (Principle of superposition).** Let \( y_1 \) and \( y_2 \) be solutions to the homogeneous equation (2). Then any linear combination \( C_1y_1 + C_2y_2 \) of \( y_1 \) and \( y_2 \), where \( C_1 \) and \( C_2 \) are constants, is also the solution to (2).

**Example 2.** Verify that \( y_1(t) = 1 \) and \( y_2(t) = t^{1/2} \) are solutions of the differential equation \( yy'' + (y')^2 = 0 \) for \( t > 0 \). Then show that \( y = c_1 + c_2 t^{1/2} \) is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 1.
**Definition**  For any two differentiable functions $y_1$ and $y_2$, the determinant
\[
W[y_1, y_2](t) = \begin{vmatrix}
  y_1(t) & y_2(t) \\
  y_1'(t) & y_2'(t)
\end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)
\]
is called the **Wronskian** of $y_1$ and $y_2$.

**Example 3.**  Find the Wronskian for the functions $e^t \sin t$, $e^t \cos t$.

**Example 4.**  If the Wronskian of $f$ and $g$ is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$. 
Definition 2. A pair of solutions \( \{y_1,y_2\} \) to \( y'' + p(t)y' + q(t)y = 0 \) on \( I \) is called fundamental solution set if
\[
W[y_1, y_2](t_0) \neq 0
\]
at some \( t_0 \in I \).

Theorem 3. (Fundamental solutions of homogeneous equations) Let \( y_1 \) and \( y_2 \) denote two solutions on \( I \) to
\[
y'' + p(t)y' + q(t)y = 0,
\]
where \( p(t) \) and \( q(t) \) are continuous on \( I \). Suppose at some point \( t_0 \in I \) these solutions satisfy
\[
W[y_1, y_2](t_0) \neq 0.
\] (3)
Then every solution to (2) on \( I \) can be expressed in the form
\[
y(t) = C_1y_1(t) + C_2y_2(t),
\] (4)
where \( C_1 \) and \( C_2 \) are constants.

Theorem 4. (Abels Theorem) If \( y_1 \) and \( y_2 \) are solutions of the differential equation
\[
y'' + p(t)y' + q(t)y = 0,
\]
where \( p \) and \( q \) are continuous on an open interval \( I \), then the Wronskian is given by
\[
W(y_1, y_2)(t) = c \exp \left[ \int p(t)dt \right],
\]
where \( c \) is a certain constant that depends on \( y_1 \) and \( y_2 \), but not on \( t \).