Chapter 8 **Numerical mathods** (brief review)

We'll try to find the solution of the initial value problem

$$\frac{dy}{dx} = f(t,x), \qquad y(x_0) = y_0 \tag{1}$$

numerically

- **The Euler of tangent line method.**

  The main idea of this method is to construct a polygonal (broken line) approximation to the solutions of the problem (1).

  Assume that the the problem (1) has a unique solution $\varphi(x)$ in some interval centered at x_0. Let h be a fixed positive number (called the *step size*) and consider the equally spaced points

$$x_n := x_0 + nh, \qquad n = 0, 1, 2, \dots$$

  The construction of values y_n that approximate the solution values $\varphi(x_n)$ proceeds as follows. At the point (x_0, y_0), the slope of the solution to (1) is given by $dy/dx = f(x_0, y_0)$. Hence, the tangent line to the curve $y = \varphi(x)$ at the initial point (x_0, y_0) is

$$y - y_0 = f(x_0, y_0)(x - x_0), \quad \text{or}$$

$$y = y_0 + f(x_0, y_0)(x - x_0).$$

  Using the tangent line to approximate $\varphi(x)$, we find that for the point $x_1 = x_0 + h$

$$\varphi(x_1) \approx y_1 := y_0 + f(x_0, y_0)(x - x_0).$$

  Next, starting at the point (x_1, y_1), we construct the line with slope equal to $f(x_1, y_1)$. If we follow the line in stepping from x_1 to $x_2 = x_1 + h$, we arrive at the approximation

$$\varphi(x_2) \approx y_2 := y_1 + f(x_1, y_1)(x - x_1).$$

  Repeating the process, we get

$$\varphi(x_3) \approx y_3 := y_2 + f(x_2, y_2)(x - x_2),$$

$$\varphi(x_4) \approx y_4 := y_3 + f(x_3, y_3)(x - x_3), \text{etc.}$$

  This simple procedure is **Euler's method** and can be summarized by the recursive formulas

$$x_{n+1} := x_0 + (n+1)h, \tag{2}$$
$$y_{n+1} := y_n + f(x_n, y_n)h, \qquad n = 0, 1, 2, \dots \tag{3}$$

The error involved in the approximation is $|e| \approx h^2$.

- **Improved Euler's method.**

  We replace $f(x_n, y_n)$ by the average of its values at the two endpoints $\frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2}$. Since $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n, y_n)$, then

$$y_{n+1} := y_n + \frac{f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))}{2}h, \qquad n = 0, 1, 2, \dots \tag{4}$$

The error involved in the approximation is $|e| \approx h^3$.

- **The Runge-Kutta method.**

$$\begin{aligned} k1 &= f(x, y) \\ k2 &= f\left(x + \frac{h}{2}, y + \frac{h}{2} \times k1\right) \\ k2 &= f\left(x + \frac{h}{2}, y + \frac{h}{2} \times k2\right) \\ k4 &= f\left(x + h, y + h \times k3\right) \\ y &= y + \frac{k1 + 2 \times k2 + 2 \times k3 + k4}{6} \\ x &= x + h \end{aligned} \tag{5}$$

The error involved in the approximation is $|e| \approx h^4$.