Chapter 1. Introduction.

- Equation that contains some derivatives of an unknown function is called a **differential equation**.
- If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter is called an **independent variable**.
- A differential equation involving only ordinary derivatives with respect to a single variable is called an **ordinary differential equations** or ODE. A differential equation involving partial derivatives with respect to more then one variable is a **partial differential equations** or PDE.

Example 1. For the following differential equations indicate independent and dependent variables.

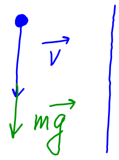
1. $y^{\mathrm{V}} + y^2 y''' + y' \sin x = 0$ $\quad$ x – independent, $y = y(x)$ – dependent, $y' = \frac{dy}{dx}$
2. $t^2 y'' + ty = 0$ $\quad$ t is independent, $y = y(t)$ dependent
3. $\underline{x'''}\sqrt{x} - 3t^2 x' + xe^t = 0$ $\quad$ $x = x(t)$ dependent, t is independent

- A differential equation that describes some physical process is often called a **mathematical model** of the process.

Examples.

1. A falling object.

   (a) Neglect the air resistance.



$\vec{v} \parallel \vec{g}$

m is the mass of the object.

net force $\vec{F} = m\vec{g}$

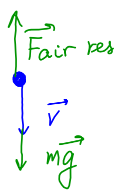
2nd Newton Law $\vec{F} = m\vec{a}$

$m\vec{a} = m\vec{g}$

component form: $\frac{dv}{dt} = a = g = 9.81\ m/sec^2$

$$\boxed{\frac{dv}{dt} = g}$$

(b) Air resistance is proportional to the velocity of the object.



$\vec{F}_{air\ res} = \gamma \vec{v}$, γ is a constant

net force: $\underbrace{m\vec{a} = \vec{F}} = \vec{F}_{air\ res} + m\vec{g}$

component form: $ma = mg - \gamma v$

or $\boxed{m\frac{dv}{dt} = mg - \gamma v}$

2. Field mice and owls. Consider a population of field mice who inhabit a certain rural area. In the absence of predators the rate of change of the mouse population is proportional to the current population.

$p(t)$ is the population of the mice

rate of change of the population is $\frac{dp}{dt}$

$\boxed{\frac{dp}{dt} = rp}$ r is a constant (growth rate)

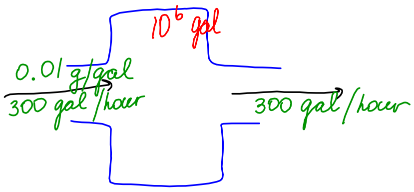
If we assume that predator rate is a constant k, $(k > 0)$, then

$\boxed{\frac{dp}{dt} = rp - k}$

Constructing math models:

1. Identify the independent and dependent variables and assign letters to represent them.
2. Choose the units of measurement for each variable.
3. Articulate the basic principle that underlines or governs the problem you are investigating.
4. Express the principle or laws from Step 3 in terms of variables from Step 1.
5. Make sure that each term in the equation has the same physical units.
6. The result of Step 4 is a single differential equation which constitutes the math model.

Example 2. A pond initially contains 10^6 gallons of water and an unknown amount of an undesirable chemical. Water containing 0.01 g of this chemical per gallon flows into the pond at a rate of 300 gal/hour. The mixture flows out at the same rate. Assume that the chemical is uniformly distributed throughout the pond. Write a differential equation for the amount of chemical in the pond at any time.



$q(t)$ is the amount of chemical in the pond @ time t.

rate of change of the amount of chemical is $\frac{dq}{dt}$

$$\frac{dq}{dt} = \boxed{\text{rate in}} - \boxed{\text{rate out}}$$

$$\boxed{\text{rate in}} = (0.01)(300) = 3$$

$$\boxed{\text{rate out}} = (300) \cdot \frac{q(t)}{10^6}$$

$$\boxed{\frac{dq}{dt} = 3 - 300\frac{q(t)}{10^6}}$$

- The **order** of a differential equation is the order of the highest-order derivatives present in equation.

  Example 3. Determine the order of the following differential equations.

  1. $yy''' - 10y'' + y^{\text{V}} = 5y^{10}$ order 5.

  2. $x'' + tx' - t^3x = \cos t$ order 2.

- A **general form** for an nth-order equation with variable x and unknown function $y = y(x)$ can be expressed as

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}\right) = 0, \tag{1}$$

  where F is a function that depends on x, y, and the derivatives of y up to the order n. We assume that the equation holds for all x in an open interval I ($a < x < b$, where a or b could be infinite).

$$\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right) \tag{2}$$

- A function $\varphi(x)$ that when substituted by y in the equation satisfies the equation for all x in the interval I is called a **solution** to the equation on I.

Example 4. Verify that $y_1(t) = e^{-3t}$ and $y_2(t) = e^t$ are solutions of the ODE

$$y'' + 2y' - 3y = 0$$

$y_1(t) = e^{-3t}$, $y_1'(t) = -3e^{-3t}$, $y_1''(t) = 9e^{-3t}$

$$\underbrace{9e^{-3t}}_{y''} + 2\underbrace{(-3e^{-3t})}_{y'} - 3\underbrace{e^{-3t}}_{y} \stackrel{?}{=} 0$$

$$9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 9e^{-3t} - 9e^{-3t} = 0$$

$$\boxed{0 = 0}$$

Example 5. Determine the values of r for which the differential equation

$$y''' - 3y'' + 2y' = 0$$

has a solution of the form $y = e^{rt}$. , $y' = re^{rt}$, $y'' = r^2e^{rt}$, $y''' = r^3e^{rt}$

$$\underbrace{r^3e^{rt}}_{y'''} - 3\underbrace{r^2e^{rt}}_{y''} + 2\underbrace{re^{rt}}_{y'} = 0$$

$$e^{rt}(r^3 - 3r^2 + 2r) = 0 \quad \text{or} \quad r^3 - 3r^2 + 2r = 0$$

$$r(r^2 - 3r + 2) = 0$$

$$r(r-1)(r-2) = 0$$

$$\boxed{r_1 = 0,\ r_2 = 1,\ r_3 = 2}$$

Example 6. Determine the values of r for which the differential equation

$$t^2y'' - 4ty' + 4y = 0$$

has a solution of the form $y = t^r$, $t > 0$.

$y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$

$$t^2\underbrace{r(r-1)t^{r-2}}_{y''} - 4t\underbrace{rt^{r-1}}_{y'} + 4\underbrace{t^r}_{y} = 0$$

$$\underbrace{t^2t^{r-2}}_{t^{2+r-2} = t^r} r(r-1) - 4\underbrace{t\,t^{r-1}}_{t^{1+r-1} = t^r} r + 4t^r = 0$$

$$t^r r(r-1) - 4t^r r + 4t^r = 0 \Rightarrow t^r(r(r-1) - 4r + 4) = 0$$

$$r(r-1) - 4r + 4 = 0$$

$$r(r-1) - 4(r-1) = 0$$

$$(r-4)(r-1) = 0$$

$$\boxed{r_1 = 1,\ r_2 = 4}$$

- By an **initial value problem** for an nth-order differential equation

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}\right) = 0$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions:

$$y(x_0) = y_0, \qquad \frac{dy}{dx}(x_0) = y_1, \ldots, \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1},$$

where $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are given constants.

- In case of a first-order equation $F\left(x, y, \frac{dy}{dx}\right) = 0$, the initial conditions reduce to the single requirement

$$y(x_0) = y_0$$

- In case of a second-order equation, the initial conditions have the form

$$y(x_0) = y_0, \qquad \frac{dy}{dx}(x_0) = y_1.$$

Example 7.

1. Show that $f(x) = (x^2 + Ax + B)e^{-x}$ is solution to

$$y'' + 2y' + y = 2e^{-x}$$

for all real numbers A and B.

$$f'(x) = (2x+A)e^{-x} - (x^2+Ax+B)e^{-x}$$
$$= (-x^2+[2-A]x + A - B)e^{-x}$$
$$f''(x) = (-2x+2-A)e^{-x} - (-x^2+[2-A]x+A-B)e^{-x}$$

$$\underbrace{(-2x+2-A)\cancel{e^{-x}} - (-x^2+[2-A]x+A-B)\cancel{e^{-x}}}_{y''} + 2\underbrace{(-x^2+[2-A]x+A-B)\cancel{e^{-x}}}_{y'} + \underbrace{(x^2+Ax+B)\cancel{e^{-x}}}_{y} \overset{?}{=} 2\cancel{e^{-x}}$$

$$-2x+2-A+x^2-2x+Ax-A+B-2x^2+4x-2Ax+2A-2B+x^2+Ax+B \overset{?}{=} 2$$

$$2 = 2$$

2. Find a solution that satisfies the initial condition $y(0) = 3$ and $y'(0) = 1$.

$$f(x) = (x^2+Ax+B)e^{-x}$$
$$f(0) = \boxed{B = 3}$$

$$f'(x) = (-x^2+[2-A]x+A-B)e^{-x}$$
$$f'(0) = A - B = 1$$
$$A = 1 + B = 1 + 3 = 4$$
$$\boxed{A = 4}$$

$$\boxed{f(x) = (x^2+4x+3)e^{-x}}$$

- An ODE is **linear** if it has format

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x),$$

  where $a_n(x)$, $a_{n-1}(x)$,...,$a_0(x)$ and $F(x)$ depend only on variable x. If an ODE is not linear, then we call it **nonlinear**.

Example 8. For each of the differential equations indicate whether it is linear or nonlinear.

1. $\underline{\ln(x)}\frac{d^2y}{dx^2} + \underline{3e^x}\frac{dy}{dx} - y\,\underline{\sin x} = 0$ linear

2. $2y'' - 3\boxed{y^2} = e^x$ nonlinear

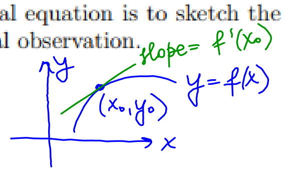
3. $\frac{d^3y}{dx^3} + \underline{(x^2-1)}y + \underline{\cos x} = 0$ linear

4. $y'' - \underline{\sin(x+y)}y' + (x^2+1)y = 0$ nonlinear

5. $y'' - \boxed{e^{xy}} = \boxed{\cos(2x+y)}$ nonlinear

Direction Fields.

One technique that is useful in graphing the solutions to a first-order differential equation is to sketch the direction field for the equation. To describe this method, we need to make a general observation.



Namely, a first-order equation

$$\frac{dy}{dx} = f(x, y)$$

specifies a slope at each point in the xy-plane where f is defined.

A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve there is called a **direction field** for the differential equation. Because the direction field gives the "flow of solutions", it facilitates the drawing of any particular solution (such as the solution to an initial value problem).

Example 9. Sketch the direction field for the equation $y' = 1 + x - y$, $y(0)=1$

$f(x,y) = 1+x-y$

$x=y=0$, $f(0,0)=1$

isoclines are curves such that the slope is the same along them.

$1+x-y=C$

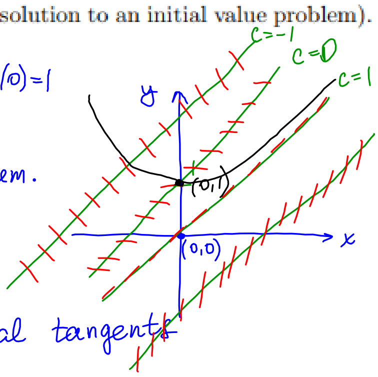
$y = 1+x-C$

$C=0$: $y=1+x$ – horizontal tangents

$C=1$: $y=x$

$C=-1$: $y=2+x$

$C=2$: $y=x-1$



Notes:

1. Integral curves cannot intersect.

2. Integral curves cannot be tangent (cannot touch).