20. Find the general solution of the system

(a) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$ $\quad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}, \; A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix}$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = 0$$

$$-2 - \lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$$\lambda_1 = -3, \; \lambda_2 = 2 \text{ — eigenvalues}$$

Corresponding eigenvectors:

$\lambda_1 = -3$. Corresponding eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of the system $(A + 3I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$4v_1 + v_2 = 0 \Rightarrow v_2 = -4v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ -4v_1 \end{pmatrix} \overset{v_1 = 1}{=} \boxed{\begin{pmatrix} 1 \\ -4 \end{pmatrix} \text{ corresponds to } \lambda_1 = -3}$$

$\lambda_2 = 2$. $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$(A - 2I)\vec{w} = \vec{0}$$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-w_1 + w_2 = 0 \Rightarrow w_1 = w_2$$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix} \overset{w_1 = 1}{=} \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = 2}$$

General solution:

$$\boxed{\vec{x}(t) = C_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}}$$

#20B).

$$\vec{x}' = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}\vec{x}. \qquad A = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}, \ tr A = -4, \det A = 4.$$

Characteristic equation: $\lambda^2 + 4\lambda + 4 = 0 = (\lambda+2)^2$

$\lambda = -2$ – repeated root.

Eigenvector: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\begin{pmatrix} -3+2 & -1 \\ 1 & -1+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad v_1 + v_2 = 0, \quad v_1 = -v_2$$

$$\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Generalized eigenvector $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$\begin{pmatrix} -3+2 & -1 \\ 1 & -1+2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \quad w_1 + w_2 = -1, \quad w_1 = -1 - w_2.$$

$$\vec{w} = \begin{pmatrix} -1 - w_2 \\ w_2 \end{pmatrix} \overset{w_2 = 0}{=\!=} \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

General solution:

$$\vec{x}(t) = e^{-2t}\left[c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left(t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)\right]$$

(0,0) is an <u>improper node, stable</u>

20c) $\vec{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \vec{x}$, $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{pmatrix}$

eigenvalues:

$$\begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (3+\lambda)(1+\lambda)+2$$

$$= 3+4\lambda+\lambda^2+2 = 0$$

$$\lambda^2+4\lambda+5=0$$

$$\lambda_1 = \frac{-4+\sqrt{16-20}}{2} = -2+i, \quad \lambda_2 = -2-i$$

Corresponding eigenvector: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$,

$$(A-(-2+i)I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} -3-(-2+i) & 2 \\ -1 & -1-(-2+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -(1+i)v_1 + 2v_2 = 0 \\ -v_1 + (1-i)v_2 = 0 \end{cases} \Rightarrow \quad v_1 = (1-i)v_2$$

$$\vec{v} = \begin{pmatrix} (1-i)v_2 \\ v_2 \end{pmatrix} \overset{v_2=1}{=\!=} \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

solutions:

$$\vec{v}e^{\lambda_1 t} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] e^{(-2+i)t} \qquad \boxed{e^{(-2+i)t} = e^{-2t}(\cos t + i \sin t)}$$

$$= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] e^{-2t}(\cos t + i \sin t)$$

$$= e^{-2t}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t + i\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin t + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t + i^2\begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t\right]$$

$$= e^{-2t}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t + i\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t\right]\right\}$$

$$= e^{-2t}\left[\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}\right]$$

General solution $\boxed{\vec{x}(t) = \left[c_1\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}\right]e^{-2t}}$

#2D.

a) $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$

<u>Variation of parameters</u>

$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$, $\operatorname{tr} A = -1$, $\det A = -6$

characteristic equation: $\lambda^2 + \lambda - 6 = 0 = (\lambda+3)(\lambda-2)$

$\lambda_1 = -3$, $\lambda_2 = 2$.

Corresponding eigenvectors: $\vec{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

General solution of the corresponding homogeneous system:

$$\vec{x}_{hom}(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

Fundamental matrix:

$$\Psi(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}, \quad \det \Psi(t) = e^{-3t}e^{2t} + 4e^{-3t}e^{2t} = 5e^{-t}.$$

$$\Psi^{-1}(t) = \frac{1}{5e^{-t}} \begin{pmatrix} e^{2t} & -e^{2t} \\ 4e^{-3t} & e^{-3t} \end{pmatrix} = \begin{pmatrix} \frac{1}{5}e^{3t} & -\frac{1}{5}e^{3t} \\ \frac{4}{5}e^{-2t} & \frac{1}{5}e^{-2t} \end{pmatrix}$$

$$\Psi^{-1}(t)\vec{g}(t) = \begin{pmatrix} \frac{1}{5}e^{3t} & -\frac{1}{5}e^{3t} \\ \frac{4}{5}e^{-2t} & \frac{1}{5}e^{-2t} \end{pmatrix} \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{5}e^{t} + \frac{2}{5}e^{4t} \\ \frac{4}{5}e^{-4t} - \frac{2}{5}e^{-t} \end{pmatrix}$$

$$\int \Psi^{-1}(t)\vec{g}(t)\,dt = \begin{pmatrix} \frac{1}{5}e^{t} + \frac{1}{10}e^{4t} + \cancel{c_1}^{\,0} \\ -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} + \cancel{c_2}^{\,0} \end{pmatrix}$$

$$\Psi(t)\int \Psi^{-1}(t)\vec{g}(t)\,dt = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix} \begin{pmatrix} \frac{1}{5}e^{t} + \frac{1}{10}e^{4t} \\ -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5}e^{-2t} + \frac{1}{10}e^{t} - \frac{1}{5}e^{-2t} + \frac{2}{5}e^{t} \\ -\frac{4}{5}e^{-2t} - \frac{4}{10}e^{t} - \frac{1}{5}e^{-2t} + \frac{2}{5}e^{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{t} \\ -e^{-2t} \end{pmatrix} = \vec{x}_p(t)$$

General solution: $\vec{x}(t) = \vec{x}_{hom}(t) + \vec{x}_p(t)$

$$= \boxed{c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} \frac{1}{2} e^{t} \\ -e^{-2t} \end{pmatrix}}$$

Laplace Transform. $\vec{x}(0) = \vec{0}$.

$$\mathcal{L}\{\vec{x}(t)\} = \vec{X}(s), \quad \mathcal{L}\{\vec{x}'(t)\} = s\vec{X}(s) - \vec{x}(0) = s\vec{X}(s)$$

$$s\vec{X}(s) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{X}(s) + \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$\vec{x}(s) = \begin{pmatrix} s-1 & -1 \\ -4 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$= \frac{1}{s^2-s+6} \begin{pmatrix} s+2 & 1 \\ 4 & s-1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s+2}{(s+2)(s^2-s+6)} + \frac{2}{(s-1)(s^2-s+6)} \\ \frac{4}{(s^2-s+6)(s+2)} - \frac{2(s-1)}{(s-1)(s^2-s+6)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s+1}{(s-1)(s-2)(s+3)} \\ \frac{-4s}{(s+2)(s+3)(s-2)} \end{pmatrix}$$

Partial fractions:

$$\frac{s+1}{(s-1)(s-2)(s+3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$= -\frac{1}{2}\frac{1}{s-1} + \frac{3}{5}\frac{1}{s-2} - \frac{1}{10}\frac{1}{s+3}$$

$$\frac{-4s}{(s+2)(s+3)(s-2)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$= -\frac{2}{s+2} + \frac{12}{5}\frac{1}{s+3} - \frac{2}{5}\frac{1}{s-2}.$$

$$\vec{x}(t) = \mathcal{L}^{-1}\{\vec{X}(s)\} = \begin{pmatrix} \mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{1}{s-1} + \frac{3}{5}\frac{1}{s-2} - \frac{1}{10}\frac{1}{s+3}\right\} \\ \mathcal{L}^{-1}\left\{-\frac{2}{s+2} + \frac{12}{5}\frac{1}{s+3} - \frac{2}{5}\frac{1}{s-2}\right\} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -\frac{1}{2}e^{t} + \frac{3}{5}e^{2t} - \frac{1}{10}e^{-3t} \\ -2e^{-2t} + \frac{12}{5}e^{-3t} - \frac{2}{5}e^{2t} \end{pmatrix}}$$

#20 b) $\vec{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \vec{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$

$A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$, $\det A = 0$, $\operatorname{tr} A = 0$.

eigenvalue $\lambda = 0$ (repeated)

corresponding eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} 4v_1 - 2v_2 &= 0 \\ 2v_1 &= v_2 \end{aligned}$$

$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Generalized eigenvector: $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} 4w_1 - 2w_2 &= 1 \\ w_1 &= \frac{1+2w_2}{4} \end{aligned}$$

$$\vec{w} = \begin{pmatrix} \frac{1+2w_2}{4} \\ w_2 \end{pmatrix} \overset{w_2=0}{=\!=} \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

General solution of the corresponding homogeneous system:

$$\vec{x}_{hom}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \right]$$

Fundamental matrix: $\Psi(t) = \begin{pmatrix} 1 & t + 1/4 \\ 2 & 2t \end{pmatrix}$, $\det \Psi(t) = -1/2$

$$\Psi^{-1}(t) = -2 \begin{pmatrix} 2t & -t - 1/4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -4t & 2t + 1/2 \\ 4 & -2 \end{pmatrix}$$

$$\int \Psi^{-1}(t) \vec{g}(t) = \begin{pmatrix} -4t & 2t + \frac{1}{2} \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{t^3} \\ -\frac{1}{t^2} \end{pmatrix} = \begin{pmatrix} -\frac{4}{t^2} - \frac{2}{t} - \frac{1}{2t^2} \\ \frac{4}{t^3} + \frac{2}{t^2} \end{pmatrix}$$

$$\int \Psi^{-1}(t)\vec{g}(t)\,dt = \begin{pmatrix} \int\left(-\frac{9}{2t^2} - \frac{2}{t}\right)dt \\ \int\left(\frac{4}{t^3} + \frac{2}{t^2}\right)dt \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{2t} - 2\ln|t| + \cancelto{0}{c_1} \\ -\frac{4}{2t^2} - \frac{2}{t} + \cancelto{0}{c_2} \end{pmatrix} = \begin{pmatrix} \frac{9}{2t} - 2\ln|t| \\ -\frac{2}{t^2} - \frac{2}{t} \end{pmatrix}$$

$$\vec{x}_p(t) = \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)\,dt$$

$$= \begin{pmatrix} 1 & t+1/4 \\ 2 & 2t \end{pmatrix}\begin{pmatrix} \frac{9}{2t} - 2\ln|t| \\ -\frac{2}{t^2} - \frac{2}{t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{2t} - 2\ln|t| - \left(\frac{2}{t^2} + \frac{2}{t}\right)\left(t + \frac{1}{4}\right) \\ \frac{9}{t} - 4\ln|t| - \frac{4}{t} - 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{2t} - 2\ln|t| - \frac{2}{t} - \frac{1}{2t^2} - 2 - \frac{1}{2t} \\ \frac{5}{t} - 4\ln|t| - 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{t} - 2\ln|t| - \frac{1}{2t^2} - 2 \\ \frac{5}{t} - 4\ln|t| - 4 \end{pmatrix}$$

$$\vec{x}(t) = \vec{x}_{hom}(t) + \vec{x}_p(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} t+1/4 \\ 2t \end{pmatrix} + \begin{pmatrix} \frac{2}{t} - 2\ln|t| - \frac{1}{2t^2} - 2 \\ \frac{5}{t} - 4\ln|t| - 4 \end{pmatrix}$$