Math 308 – 201 Quiz 13 7/31/2018

NAME (print): ______________________

No credit for unsupported answers will be given. Clearly indicate your final answer

Given the system

$$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 4x_1 - 2x_2 \end{cases}$$

1. Write the system in matrix notation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
2. Find the eigenvalues and eigenvectors for the coefficient matrix $\mathbf{A}$.
3. What is the general solution of the system?
4. Sketch its phase portrait.

1pt

1. $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$, $\operatorname{tr} A = -1$, $\det A = -6$

2. $\begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0$

$\lambda^2 + \lambda - 6 = 0$ 2pt

$(\lambda+3)(\lambda-2) = 0$, $\lambda_1 = -3$, $\lambda_2 = 2$ eigenvalues (1pt, 1pt)

$\lambda_1 = -3$: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4v_1 + v_2 = 0 \rightarrow v_2 = -4v_1$

$\vec{v} = \begin{pmatrix} v_1 \\ -4v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$\boxed{\vec{x}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \text{ corresponds to } \lambda = -3}$ 1pt

$\lambda_2 = 2$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

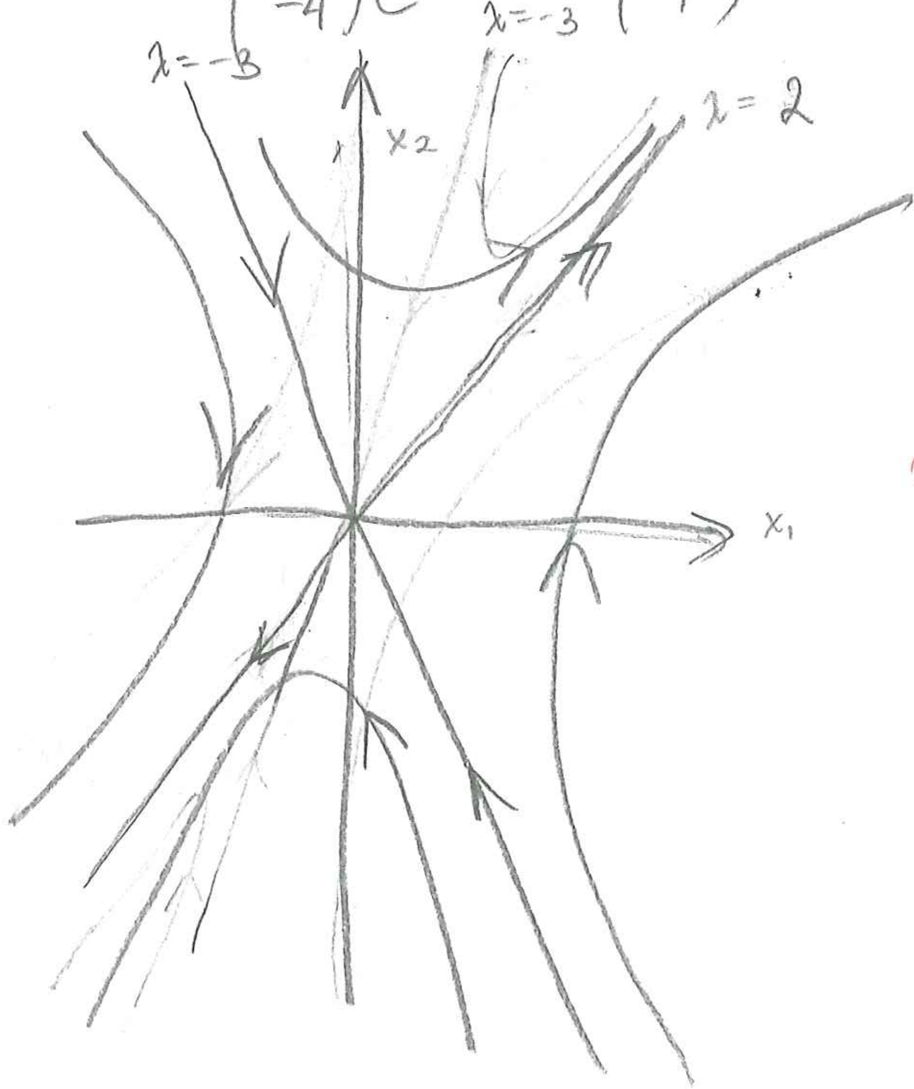
$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow w_1 = w_2$

$\boxed{\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda = 2}$ 1pt

3. General solution

$$\vec{x} = C_1\begin{pmatrix}1\\-4\end{pmatrix}e^{-3t} + C_2\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$$ 1 pt

4.



2pts.

Math 308 – 201 Quiz 14 8/1/2018

NAME (print): ______________________

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Given the system

$$\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 4x_1 - x_2 \end{cases}$$

1. Write the system in matrix notation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
2. Find the eigenvalues and eigenvectors for the coefficient matrix $\mathbf{A}$.
3. What is the general solution of the system? Make sure it includes only real-valued functions.
4. Sketch its phase portrait.

$$\vec{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x}$$ 1pt

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$tr(A) = 2$, 1pt $det(A) = -3 + 8 = 5$ 1pt

characteristic equation: $\lambda^2 - 2\lambda + 5 = 0$. 1pt

$$\lambda_1 = \frac{2+\sqrt{4-20}}{2} = \frac{2+\sqrt{-16}}{2} = \frac{2+4i}{2} = 1+2i$$ 1pt

$$\lambda_2 = \bar{\lambda}_1 = 1-2i$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : \begin{pmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-2i)v_1 - 2v_2 = 0 \Rightarrow v_2 = (1-i)v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ (1-i)v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \overset{v_1=1}{=} \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$ 1pt

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$$\vec{x}(t) = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+2i)t} = e^t \begin{pmatrix} 1 \\ 1-i \end{pmatrix} (\cos 2t + i \sin 2t)$$

$$= e^t \begin{pmatrix} \cos 2t + i \sin 2t \\ \cos 2t + i \sin 2t - i \cos 2t + \sin 2t \end{pmatrix}$$

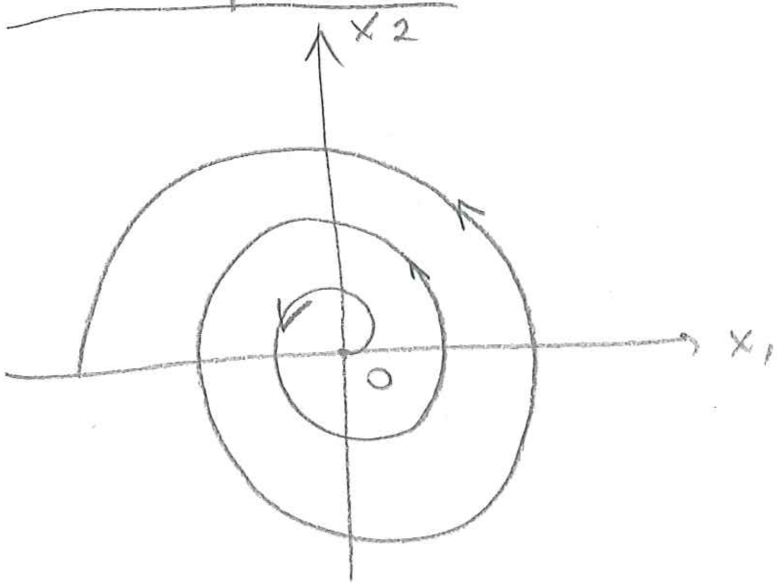
1pt

$$= e^t \left[\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix} \right]$$

General solution 1pt 1pt

$$\vec{x}(t) = e^t \left[c_1 \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix} \right]$$

Phase portrait:



1pt

Math 308 – 201 Quiz 15 8/2/2018

NAME (print): ______________________

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Given the system

$$\begin{cases} x_1' = x_1 - 4x_2 \\ x_2' = 4x_1 - 7x_2 \end{cases}$$

1. Write the system in matrix notation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
2. Find the eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$.
3. Find a generalized eigenvector.
4. What is the general solution of the system?
5. Sketch its phase portrait.

1 $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ 1pt

$tr(A) = -6$, $\det(A) = -7+16 = 9$

2a charact. poly. $\lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)^2 = 0$ 2pts

$\lambda = -3$ repeated 1pt

2b eigenvector: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $(A+3I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = v_2$$

$\boxed{\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda = -3}$ 2pts

3 Generalized eigenvector: $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $(A+3I)\vec{w} = \vec{v}$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow 4w_1 - 4w_2 = 1$$

$$w_2 = \frac{1}{4}(1 - 4w_1)$$

$$\vec{w} = \begin{pmatrix} w_1 \\ -\frac{1}{4}(1-4w_1) \end{pmatrix} \overset{w_1=0}{=} \begin{pmatrix} 0 \\ -\frac{1}{4} \end{pmatrix} \text{ or } \vec{w} = \begin{pmatrix} +\frac{1}{4} \\ 0 \end{pmatrix}$$

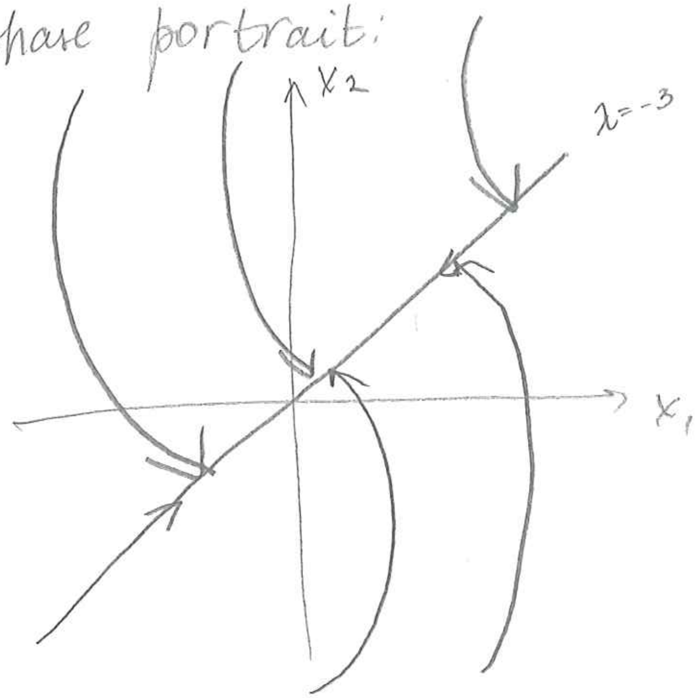
[2pts]

4 General solution

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 t \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + \vec{w} e^{-3t}}$$

1pt

5 phase portrait:



countrclockwise.

1pts.

Math 308 – 201 Quiz 16 8/3/2018

NAME (print): ______________________

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Given the system

$$\begin{cases} x_1' = -2x_1 + x_2 + 2e^{-t} \\ x_2' = x_1 - 2x_2 + 3t \end{cases}, \quad \vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

1. Find the form of a particular solution using the method of undetermined coefficients. (DO NOT solve for unknown constants).

2. Find a particular solution using variation of parameters $\mathbf{x}_p(t) = \Psi(t)\int \Psi^{-1}(t)\mathbf{g}(t)dt$

3. Find the Laplace Transform of $\mathbf{x}(t)$: $\mathcal{L}\{\mathbf{x}(t)\} = \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{G}(s)$.

characteristic polynomial: $\lambda^2 + 4\lambda + 3 = 0$, $(\lambda_1 = -3, \lambda_2 = -1)$ 1 pts

eigenvectors: $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1. $\boxed{\vec{x}_p = \vec{a}e^{-t} + \vec{b}te^{-t} + \vec{c}t + \vec{d}}$ 1pt

2. $\Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$ 1pt, $\Psi^{-1}(t) = \frac{1}{2e^{-4t}}\begin{pmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix}$ 1pt

$$\Psi^{-1}(t)\vec{g} = \begin{pmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} - \frac{3}{2}te^{3t} \\ 1 + \frac{3}{2}te^{t} \end{pmatrix}$$ 1pt

$\int te^{3t}dt = \frac{1}{3}\int t\,d(e^{3t}) = \frac{1}{3}(te^{3t} - \frac{1}{3}e^{3t}) + \cancel{c_1}^{\,0}$

$\int te^{t}dt = te^{t} - e^{t} + \cancel{c_2}^{\,0}$

$$\int \Psi^{-1}\vec{g}\,dt = \begin{pmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} \end{pmatrix}$$

$$\boxed{\vec{x}_p = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} \end{pmatrix}}$$ 1pt

3. $\vec{G}(s) = \mathcal{L}\{\vec{g}(t)\} = \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$ 2pts

$$sI - A = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}, \det(sI-A) = (s+2)^2 + 1 = s^2+4s+3$$

$$(sI-A)^{-1} = \begin{pmatrix} \frac{s+2}{s^2+4s+3} & +\frac{1}{s^2+4s+3} \\ +\frac{1}{s^2+4s+3} & \frac{s+2}{s^2+4s+3} \end{pmatrix}$$ 1pt

$$\vec{X}(s) = \begin{pmatrix} \frac{s+2}{s^2+4s+3} & \frac{1}{s^2+4s+3} \\ \frac{1}{s^2+4s+3} & \frac{s+2}{s^2+4s+3} \end{pmatrix} \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2(s+2)}{(s^2+4s+3)(s+1)} + \frac{3}{(s^2+4s+3)s^2} \\ \frac{2}{(s+1)(s^2+4s+3)} + \frac{3(s+2)}{(s^2+4s+3)s^2} \end{pmatrix}$$ 1pt