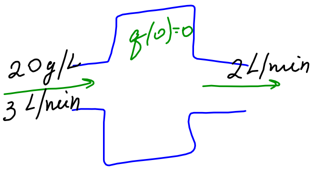
over sections 1.1–1.3, 2.1–2.6, 3.1 – 3.4

1. A large tank initially contains 10 L of fresh water. A brine containing 20 g/L of salt flows into the tank at the rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the rate of 2 L/min. Determine the concentration of salt in the tank as a function of time.



$q(t)$ is the mass @ tank @ time t

$$\frac{dq}{dt} = \text{rate in} - \text{rate out}$$

$$= 20(3) - (2)\frac{q(t)}{10+(3-2)t}$$

$$\frac{dq}{dt} = 60 - \frac{2q}{10+t}, \quad q(0)=0$$

$$\frac{dq}{dt} + \frac{2}{10+t}q = 60$$

linear, $P(t) = \frac{2}{10+t}$, $Q(t) = 60$

Integrating factor μ: $\frac{d\mu}{dt} = \frac{2}{10+t}\mu$ separable

$$\int \frac{d\mu}{\mu} = \int \frac{2}{10+t}\,dt$$

$$\ln|\mu| = 2\ln|10+t| \Rightarrow \mu(t) = (10+t)^2$$

$$\mu(t)q(t) = \int 60\,\mu(t)\,dt$$

$$(10+t)^2 q(t) = 60\int (10+t)^2\,dt$$

$$\frac{(10+t)^2 q(t)}{(10+t)^2} = \frac{20(10+t)^3 + C}{(10+t)^2}$$

$$q(t) = 20(10+t) + \frac{C}{(10+t)^2}$$

$$q(0) = 200 + \frac{C}{100} = 0 \Rightarrow C = -20000$$

$$q(t) = 20(10+t) - \frac{20000}{(10+t)^2} \quad \text{- mass}$$

$$\boxed{\frac{q(t)}{10+t} = 20 - \frac{20000}{(10+t)^3}} \quad \text{concentration}$$

2. An object with temperature 150° is placed in a freezer whose temperature is 30°. Assume that the temperature of the freezer remains essentially constant.

(a) If the object is cooled to 120° after 8 min, what will its temperature be after 18 min?

(b) When will its temperature be 60°?

$T(t)$ is the temperature of the object @ time t.

$$\boxed{\frac{dT}{dt} = k(M-T)}, \quad M \text{ is the outside temperature. } k \text{ is an unknown constant}$$

$M = 30$

$T(0) = 150$, $T(8) = 120$, $T(18) = ?$

time t such that $T(t) = 60$

$$\frac{dt\,\frac{dT}{dt}}{T-30} = \frac{-k(30+T)\,dt}{T-30} \Rightarrow \int \frac{dT}{T-30} = \int -k\,dt$$

$$\ln|T-30| = -kt + C$$

$$T - 30 = C_1 e^{-kt}, \quad C_1 = e^C$$

$$T = 30 + C_1 e^{-kt}$$

$$T(0) = 30 + C_1 = 150 \Rightarrow T(t) = 30 + 120e^{-kt}$$

$$C_1 = 120$$

$$T(8) = 30 + 120e^{-8k} = 120$$

$$120e^{-8k} = 90$$

$$e^{-8k} = \frac{9}{12} = \frac{3}{4} \Rightarrow -8k = \ln\frac{3}{4} \Rightarrow k = -\frac{1}{8}\ln\frac{3}{4}$$

$$\boxed{T(t) = 30 + 120e^{\frac{t}{8}\ln\frac{3}{4}}}$$

$$T(18) = 30 + 120e^{\frac{18}{8}\ln\frac{3}{4}}$$

Find t such that $T(t) = 30 + 120e^{\frac{t}{8}\ln\frac{3}{4}} = 60$

$$120e^{\frac{t}{8}\ln\frac{3}{4}} = 30$$

$$e^{\frac{t}{8}\ln\frac{3}{4}} = \frac{1}{4}$$

$$\frac{t}{8}\ln\frac{3}{4} = \ln\frac{1}{4}$$

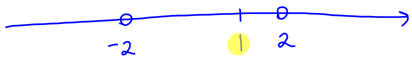
$$\boxed{t = 8\,\frac{\ln\frac{1}{4}}{\ln\frac{3}{4}}}$$

3. Determine (without solving the problem) an interval in which the solution to the initial value problem

$$\underbrace{(4-t^2)y' + 2ty = 3t^2}_{(4-t^2)}, \qquad y(1) = -3$$

is certain to exist.

$$y' + \frac{2t}{4-t^2}\,y = \frac{3t^2}{4-t^2}, \qquad 4-t^2 \neq 0, \quad t \neq \pm 2$$



$$\boxed{(-2,2)}$$

$\sqrt{t} \Rightarrow t \geq 0$	$\cot t = \frac{\cos t}{\sin t} \Rightarrow t \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$
$\ln t \rightarrow t > 0$	$\tan t = \frac{\sin t}{\cos t} \Rightarrow t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$
$\frac{1}{\ln t} \rightarrow t > 0, \ t \neq 1$	

$y' + p(t)\,y = g(t)$

4. Solve the initial value problem

$$y' = \frac{t^2}{1+t^3}, \qquad y(0) = y_0$$

and determine how the interval in which the solution exists depends on the initial value y_0.

$g(t) = f(y,t) = \dfrac{t^2}{1+t^3}$

continuous for all $t \neq -1$

$-1 \qquad 0$

$\boxed{(-1,\infty)}$ solution on

$\dfrac{dy}{dt} = \dfrac{t^2}{1+t^3} \Rightarrow dy = \dfrac{t^2}{1+t^3}\,dt$

$y = \frac{1}{3}\ln|1+t^3| + C$

$y(0) = \frac{1}{3}\ln 1 + C = C = y_0$

solution of the initial value problem:

$\boxed{y = \frac{1}{3}\ln|1+t^3| + y_0}$ exists for all y_0

5. Solve the following initial value problem

$$\sqrt{y}dt + (1+t)dy = 0 \quad y(0) = 1.$$

not linear

not exact

separable.

$$\sqrt{y}\, dt = -(1+t)dy$$

$$\int \frac{dy}{\sqrt{y}} = -\int \frac{dt}{1+t}$$

$$\frac{y^{1/2}}{1/2} = -\ln|1+t| + C$$

$$2\sqrt{y} = C - \ln|1+t|$$

$$\sqrt{y} = \frac{C - \ln|1+t|}{2}$$ plug in $y=1$ and $t=0$

$$1 = \frac{C - \ln 1}{2}$$ or $C = 2$

$$\boxed{\sqrt{y} = \frac{2 - \ln|1+t|}{2}}$$ solution of the initial value problem.

6. Find the general solution of the equation

$$(t^2 - 1)y' + 2ty + 3 = 0$$ linear

$$y' + \frac{2t}{t^2-1} y + \frac{3}{t^2-1} = 0$$

$$p(t) = \frac{2t}{t^2-1}, \quad g(t) = -\frac{3}{t^2-1}$$

$$\frac{d\mu}{dt} = \frac{2t}{t^2-1}\mu \Rightarrow \mu = t^2-1$$

$$(t^2-1)y = \int -\frac{3}{t^2-1}(t^2-1)\,dt = -3t + C$$

$$\boxed{y = -\frac{3t}{t^2-1} + \frac{C}{t^2-1}}$$

$$(t^2-1)y' + 2ty + 3 = 0$$

7. Given the differential equation

$$\frac{dy}{dt} = 7y - y^2 - 10$$

(a) Find the equilibrium solutions

(b) Sketch the phase line and determine whether the equilibrium solutions are stable, unstable, or semistable

(c) Graph some solutions

(d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$, where $-\infty < y_0 < \infty$, find the limit of $y(t)$ when $t \to \infty$

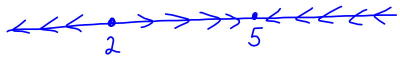
(e) Solve the equation

7a) $7y - y^2 - 10 = 0 \Rightarrow y^2 - 7y + 10 = 0.$

$(y-5)(y-2) = 0$

$y_1 = +5,\ y_2 = 2$

7b) $f(y) = 7y - y^2 - 10$



$f(0) = -10 < 0$

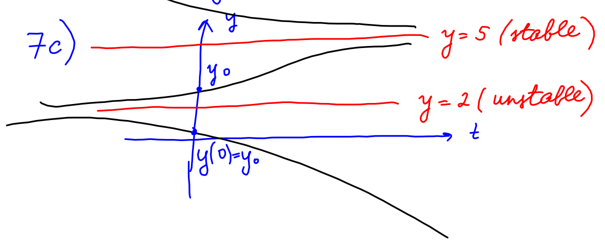
$f(3) = 7(3) - 9 - 10 > 0$

$f(6) = 42 - 36 - 10 < 0$

$y = 2$ unstable

$y = 5$ stable

7c)



7d) $y(0) = y_0.$

$y_0 < 2$	$2 < y_0 < 5$	$y_0 > 5$
$\lim_{t\to\infty} y(t) = -\infty$	$\lim_{t\to\infty} y(t) = 5$	$\lim_{t\to\infty} y(t) = 5$
$\lim_{t\to-\infty} y(t) = 2$	$\lim_{t\to-\infty} y(t) = 2$	$\lim_{t\to-\infty} y(t) = \infty$

$$\frac{dy}{dt} = 7y - y^2 - 10$$

$$\frac{dy}{dt} = -(y-5)(y-2)$$

$$\frac{dy}{(y-5)(y-2)} = -dt$$

$$\frac{1}{(y-5)(y-2)} = \frac{1}{3}\left[\frac{1}{y-5} - \frac{1}{y-2}\right]$$

$$\frac{1}{3}\left[\ln|y-5| - \ln|y-2|\right] = -t + C$$

$$\boxed{\ln\left|\frac{y-5}{y-2}\right| = -3t + C_1}$$

$(uvw)' = u'vw + uv'w + uvw'$

8. Solve the initial value problem

$$\underbrace{(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x)}_{M}dx + \underbrace{(xe^{xy}\cos(2x) - 3)}_{N}dy = 0, \qquad y(0) = -1$$

$\frac{\partial M}{\partial y} = e^{xy}\cos(2x) + y\,e^{xy}(x)\cos(2x) - 2e^{xy}(x)\sin(2x)$

match

$\frac{\partial N}{\partial x} = e^{xy}\cos(2x) + x\,e^{xy}y\cos(2x) + xe^{xy}(-2)\sin(2x)$

exact

$F(x,y): \begin{cases} \frac{\partial F}{\partial x} = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x \\ \int\frac{\partial F}{\partial y}dy = \int\left[xe^{xy}\cos(2x) - 3\right]dy \end{cases}$

$\int e^{xy}dy = \frac{1}{x}e^{xy} + C$

$F(x,y) = \frac{x}{x}e^{xy}\cos 2x - 3y + g(x)$

$F(x,y) = e^{xy}\cos 2x - 3y + g(x)$

$\frac{\partial F}{\partial x} = e^{xy}\cancel{(y)\cos 2x} + \cancel{e^{xy}(-2)\sin(2x)} + g'(x) = \cancel{ye^{xy}\cos(2x)} - \cancel{2e^{xy}\sin(2x)} + 2x$

$g'(x) = 2x$ or $g(x) = x^2 + C$

$F(x,y) = e^{xy}\cos 2x - 3y + x^2 + C$

General solution: $\boxed{e^{xy}\cos 2x - 3y + x^2 + C = 0}$

9. Find an integrating factor for the equation

$$\underbrace{(3xy + y^2)}_{M} + \underbrace{(x^2 + xy)}_{N} y' = 0$$

and then solve the equation.

$$\frac{M_y - N_x}{N} = \frac{(3x+2y)-(2x+y)}{x^2+xy} = \frac{x+y}{x(x+y)} = \frac{1}{x} \text{ depends on } x \text{ only}$$

Integrating factor $\mu(x)$: $\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$

$$\frac{d\mu}{dx} = \frac{\mu}{x}$$

$$\frac{d\mu}{\mu} = \frac{dx}{x} \Rightarrow \ln|\mu| = \ln|x| \Rightarrow \mu(x) = x$$

$$x(3xy+y^2) + x(x^2+xy)y' = 0, \quad y' = \frac{dy}{dx}$$

$$\underbrace{(3x^2y + xy^2)}_{M(x,y)}dx + \underbrace{(x^3+x^2y)}_{N(x,y)}dy = 0$$

$$\frac{\partial M}{\partial y} = 3x^2 + 2xy \qquad \frac{\partial N}{\partial x} = 3x^2 + 2xy \qquad \text{exact}.$$

$F(x,y)$: $\begin{cases} \frac{\partial F}{\partial x} = 3x^2y + xy^2 \\ \int \frac{\partial F}{\partial y} dy \int [x^3 + x^2 y]\, dy \end{cases} \Rightarrow F(x,y) = x^3 y + \frac{x^2y^2}{2} + g(x)$

$$\frac{\partial F}{\partial x} = \cancel{3x^2y} + \cancel{2xy^2} + g'(x) = \cancel{3x^2y} + \cancel{xy^2}$$

$$g'(x) = 0$$

$$g(x) = C$$

$$F(x,y) = x^3y + \frac{x^2y^2}{2} + C$$

General solution: $\boxed{x^3y + \frac{x^2y^2}{2} + C = 0}$

10. Solve the equation/initial value problem

(a) $6y'' - 5y' + y = 0$, $y(0) = 4$, $y'(0) = 0$

(b) $4y'' - 12y' + 9y = 0$

(c) $y'' + 4y' + 5y = 0$, $y(0) = 0$, $y'(0) = 1$

a) $6y'' - 5y' + y = 0$, $y(0)=4$, $y'(0)=0$

auxiliary equation: $y'' \to r^2$, $y' \to r$, $y \to 1$

$6r^2 - 5r + 1 = 0$

$r_1 = \frac{5+\sqrt{25-24}}{12} = \frac{1}{2}$

$r_2 = \frac{5-1}{12} = \frac{4}{12} = \frac{1}{3}$

General solution: $y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{\frac{t}{3}}$

plug into the initial conditions

$y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{\frac{t}{3}}$ | $y(0) = c_1 + c_2 = 0$

$y'(t) = \frac{c_1}{2} e^{\frac{t}{2}} + \frac{c_2}{3} e^{\frac{t}{3}}$ | $y'(0) = \frac{c_1}{2} + \frac{c_2}{3} = 1$

$\begin{cases} c_1 + c_2 = 0 \\ 3c_1 + 2c_2 = 6 \end{cases} \Rightarrow c_2 = -c_1$

$3c_1 - 2c_1 = 6 \Rightarrow c_1 = 6, \; c_2 = -6$

$\boxed{y(t) = 6e^{\frac{t}{2}} - 6e^{\frac{t}{3}}}$

b) $4y'' - 12y' + 9y = 0$

$4r^2 - 12r + 9 = 0$

$(2r-3)^2 = 0$

$r = \frac{3}{2}$ repeated

$\boxed{y(t) = (c_1 + c_2 t)e^{\frac{3}{2}t}}$

c) $y'' + 4y' + 5y = 0$, $y(0) = 0$, $y'(0) = 1$

$r^2 + 4r + 5 = 0$

$r_1 = \frac{-4+\sqrt{16-20}}{2} = \frac{-4+\sqrt{-4}}{2} = \frac{-4+2i}{2}$

$r_1 = -2+i$

$Re(r_1) = -2$, $Im(r_1) = 1$

General solution is

$\boxed{y(t) = e^{-2t}[c_1 \cos t + c_2 \sin t]}$

$y(0) = \boxed{c_1 = 0}$

$y'(t) = -2e^{-2t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(-c_1 \sin t + c_2 \cos t)$

$y'(0) = -2c_1 + c_2 = 1 \Rightarrow c_2 = 1$

$\boxed{y(t) = e^{-2t} \sin t}$

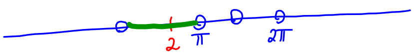
11. Find the interval(s) on which the solution of the initial value problem

$$x^3 y'' + \frac{x}{\sin x} y' - \frac{2}{x-5} y = 0, \quad y(2) = 6, \quad y'(2) = 7$$

is certain to exist.

$$y'' + \frac{x}{x^3 \sin x} y' - \frac{2}{x^3(x-5)} y = 0$$

$x \neq 0$, $x \neq 5$, $\sin x \neq 0 \Rightarrow x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$



$\boxed{(0, \pi)}$

12. Find the Wronskian of two functions $y_1(x) = x + 2x^2$ and $y_2(x) = 2^x$.

$y_1' = 4x \qquad y_2' = 2^x \ln 2$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W[y_1, y_2] = \begin{vmatrix} x+2x^2 & 2^x \\ 4x & 2^x \ln 2 \end{vmatrix} = \boxed{2^x \ln 2 (x+2x^2) - 2^x(4x)}$$