Section 2.1 **Linear Equations.**

Definition. A **linear first-order equation** is an equation that can be expressed in the form

$$\frac{a_1(x)\frac{dy}{dx} + a_0(x)y}{a_1(x)} = \frac{b(x)}{a_1(x)},$$

where $a_0(x)$, $a_1(x)$, $b(x)$ depend only on x.

We will assume that $a_0(x)$, $a_1(x)$, $b(x)$ are continuous functions of x on an interval I.

For now, we are interested in those linear equations for which $a_1(x)$ is never zero on I. In that case we can rewrite the equation in the standard form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x),}$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$ are continuous on I.

To solve the first order linear equation:

1. Write the equation $a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$ in the standart form $\frac{dy}{dx} + P(x)y = Q(x)$.
2. Find the integrating factor $\mu(x)$ solving differential equation

$$\mu(x):\ \boxed{\frac{d\mu}{dx} - P(x)\mu = 0.}$$

3. Integrate the equation

$$\frac{d}{dx}[\mu y] = \mu Q(x)$$

and solve for y by dividing by $\mu(x)$.

The solution

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right]$$

where C is a constant, to the equation is called the **general solution**.

want to find $\mu(x)$ such that

$$\mu(x)\left[\frac{dy}{dx} + P(x)y\right] = \frac{d}{dx}(\mu \cdot y)$$

Example 1. Find the general solution to the following equations

1. $xy' - y = -\ln x$

$$\frac{xy' - y}{x} = \frac{-\ln x}{x}$$

$$y' - \frac{y}{x} = -\frac{\ln x}{x}$$

$$y' - \frac{1}{x}\cdot y = -\frac{\ln x}{x}$$

$$P(x) = -\frac{1}{x}, \quad Q(x) = -\frac{\ln x}{x}$$

Integrating factor $\mu(x)$: $\quad \frac{d\mu}{dx} - P(x)\mu = 0$ or $\frac{d\mu}{dx} = P(x)\mu$

$$dx\,\frac{d\mu}{dx} = -\frac{1}{x}\mu\, dx \quad \text{separable}$$

$$\int \frac{d\mu}{\mu} = -\int \frac{dx}{x}$$

$$\ln|\mu| = -\ln|x| + \cancel{C}^{\,0} \Rightarrow \ln|\mu| = \ln|x|^{-1}$$

$$\boxed{\mu = \frac{1}{x}}$$

$$\frac{d}{dx}(\mu y) = Q\mu$$

$$\int \frac{d}{dx}\left(\frac{y}{x}\right)dx = \int -\frac{\ln x}{x^2}\,dx \Rightarrow \quad \frac{y}{x} = -\int \frac{\ln x}{x^2}\,dx$$

by parts $\quad u = \ln x \quad v' = -\frac{1}{x^2}$

$\quad u' = \frac{1}{x} \quad v = \frac{1}{x}$

$$\frac{y}{x} = \frac{\ln x}{x} - \int \frac{1}{x^2}\,dx$$

$$\frac{y}{x} = \frac{\ln x}{x} + \frac{1}{x} + C \quad \text{- solve for } y$$

$$\boxed{y = \ln x + 1 + Cx}$$

Example 1. Find the general solution to the following equations

1. $xy' - y = -\ln x$

2. $y' - 2y = t^2 e^{2t}$

– standard form

$P(t) = -2, \quad Q(t) = t^2 e^{2t}$

Integrating factor $\mu(t)$:

$$\frac{d\mu}{dt} = P(t)\mu$$

$$\frac{\frac{d\mu}{dt}}{\mu} dt = \frac{-2\mu}{\mu} dt$$

$$\int \frac{d\mu}{\mu} = -\int 2\,dt$$

$$\ln|\mu| = -2t + \cancel{c}^{\,0}$$

$$\boxed{\mu(t) = e^{-2t}}$$

$$\frac{d}{dt}(\mu y) = Q\mu$$

$$\frac{d}{dt}\left(ye^{-2t}\right) = t^2 e^{2t} \cdot e^{-2t}$$

$$\int \frac{d}{dt}\left(ye^{-2t}\right) dt = \int t^2\,dt$$

$$y\,e^{-2t} = \frac{t^3}{3} + C$$

$$\boxed{y(t) = \frac{t^3}{3} e^{2t} + Ce^{2t}}$$

Example 2. Solve the initial value problem

1. $\underbrace{dy - y\,dx - 2xe^x\,dx}_{dx} = \underset{dx}{0}, \quad y(0) = e - 2$

$\frac{dy}{dx} - y - 2xe^x = 0$ or $\frac{dy}{dx} - y = 2xe^x$

$P(x) = -1, \quad Q(x) = 2xe^x$

Integrating factor $\mu(x)$: $\frac{d\mu}{dx} = P(x)\mu$

$\frac{\frac{d\mu}{dx}dx}{\mu} = \frac{-\mu\,dx}{\mu}$

$\int \frac{d\mu}{\mu} = -\int dx \Rightarrow \ln|\mu| = -x + \cancel{c}^{\,0}$

$\mu(x) = e^{-x}$

$\frac{d}{dx}(y\mu) = \mu \cdot Q$

$y e^{-x} = \int \overbrace{2xe^x}^{Q} \cdot \overbrace{e^{-x}}^{\mu}\,dx = x^2 + C$

$\boxed{y(x) = x^2e^x + Ce^x}$

2. $ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, t > 0$