Chapter 2. First Order Differential Equations
Section 2.2 **Separable Equations**

$$y' = \frac{dy}{dx} = f(x, y)$$

Sometimes a function $f(x, y)$ can be represented as a product of two functions, one of which depens ONLY on x, another depens ONLY on y, or $f(x, y) = g(x)h(y)$. Then

$$\frac{dy}{dx} = g(x)h(y).$$

Definition. A differential equation $y' = f(x, y)$ is called **separable** if it can be written in the form

$$M(x)dx + N(y)dy = 0$$

Example 1. Determine whether the given equation is separable.

1. $(t - 2y)^2 y' = 2$ $\boxed{\text{no}}$

2. $y^4 e^y + (t^3 + 1)y' = y'(t^3 + 1)e^{2y}$ solve for y'.

   $y'(t^3+1) - y'(t^3+1)e^{2y} = -y^4 e^y \Rightarrow \frac{y'(t^3+1)[1-e^{2y}]}{(t^3+1)(1-e^{2y})} = \frac{-y^4e^y}{(t^3+1)(1-e^{2y})} \Rightarrow y' = \frac{-y^4e^y}{(t^3+1)(1-e^{2y})}$ $\boxed{\text{yes}}$

3. $yx \ln x dx - \sqrt{y}dy + x \ln x dx = 0$

   $x\ln x dx(y+1) = \sqrt{y}\,dy \Rightarrow x\ln x\, dx = \frac{\sqrt{y}}{y+1}\,dy$ $\boxed{\text{yes}}$

4. $y' = \cot^2\left(\frac{x}{2} + y - 1\right) + \frac{1}{2}$ $\boxed{\text{no}}$

How to solve a separable equation?

$$y' = g(x)h(y)$$

$$y' = \frac{dy}{dx}$$

$$dx\left(\frac{\frac{dy}{dx}}{h(y)}\right) = \left(\frac{g(x)h(y)}{h(y)}\right)dx, \quad h(y) \neq 0$$

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

$$\boxed{H(y) = G(x) + C}, \quad C \text{ is a constant}$$

general solution of the equation.

$$[H(y)]' = \frac{1}{h(y)}$$

$$[G(x)]' = g(x)$$

Check separately if zeroes of $h(y)$ are solutions of the equation.

Example 2. Solve the equations/initial value problems:

1. $xydx + (x+1)dy = 0$

$$\frac{(x+1)\,dy}{(x+1)y} = \frac{-xy\,dx}{(x+1)y}$$

$$\int \frac{dy}{y} = -\int \frac{(x+1)-1}{x+1}dx$$

$$\ln|y| = -\int \left[\frac{x+1}{x+1} - \frac{1}{x+1}\right]dx$$

$$\ln|y| = -\int\left[1 - \frac{1}{x+1}\right]dx \Rightarrow$$

$$\boxed{\ln|y| = -\left[x - \ln|1+x|\right] + C} \quad \text{implicit solution}$$

$$\cancel{e^{\ln|y|}}\; y = e^{-[x-\ln|1+x|]+C}$$

$$y = e^{-x+\ln|1+x|+C}$$

$$= e^{-x}\cdot \cancel{e^{\ln|1+x|}}\,(1+x) \cdot \cancel{e^{C}}\,C_1, \quad C_1 = e^{C} \text{ constant}$$

$$\boxed{y = C_1 e^{-x}(1+x)} \quad \text{explicit solution}$$

Check $\boxed{y=0}$ solution, $dy = 0$

$$x(0)\,dx + (x+1)(0) = 0$$

and $\boxed{x=-1}$ another solution. $dx = 0$

$$(-1)y(0) + (1-1)dy = 0$$

2. $(x^2-1)y' + 2xy^2 = 0, \quad y(0) = 1$

$$y' = \frac{dy}{dx}$$

$$\frac{(x^2-1)\frac{dy}{dx}\,dx}{y^2(x^2-1)} = \frac{-2xy^2\,dx}{y^2(x^2-1)}$$

$$\int\frac{dy}{y^2} = -\int\frac{2x\,dx}{x^2-1}$$

$$-\frac{1}{y} = -\ln|x^2-1| + C$$

$$\boxed{y = \frac{1}{\ln|x^2-1| - C}} \quad \text{general solution}$$

plug in $x = 0$

$$y(0) = \frac{1}{\ln|-1| - C} = \frac{1}{\ln 1 - C} = -\frac{1}{C} = 1 \Rightarrow C = -1$$

$$\boxed{y = \frac{1}{\ln|x^2-1| + 1}} \quad \text{the solution of the initial value problem}$$

3. $xy\,dx - \sqrt{x^2+1}\ln^2 y\,dy = 0$, $y>0$

$$\frac{\sqrt{x^2+1}\,\ln^2 y\,dy}{y\sqrt{x^2+1}} = \frac{xy\,dx}{y\sqrt{x^2+1}}$$

$$\int \frac{\ln^2 y}{y}\,dy = \int \frac{x}{\sqrt{x^2+1}}\,dx$$

$u = \ln y$ $\qquad u = x^2+1$

$$\frac{\ln^3 y}{3} = \frac{1}{2}\frac{(x^2+1)^{1/2}}{1/2} + C \Rightarrow \boxed{\frac{\ln^3 y}{3} = (x^2+1)^{1/2} + C}$$ implicit solution

4. $x\cos^2 y\,dx - e^x \sin 2y\,dy = 0,\quad y(0) = 0$

$$\frac{e^x \sin 2y\,dy}{e^x\cos^2 y} = \frac{x\cos^2 y\,dx}{e^x\cos^2 y}$$

$\boxed{\sin 2y = 2\sin y\cos y}$

$$\frac{\sin 2y}{\cos^2 y}\,dy = x e^{-x}\,dx$$

$\boxed{\int uv'dx = uv - \int vu'dx}$

$$\frac{2\sin y\cos y}{\cos^2 y}\,dy = xe^{-x}\,dx$$

by parts $\quad u = x \quad v' = e^{-x}$

$\quad u' = 1 \quad v = -e^{-x}$

$$2\int\frac{\sin y}{\cos y}\,dy = \int xe^{-x}\,dx$$

$$-2\ln|\cos y| = -xe^{-x} + \int e^{-x}\,dx$$

$$+2\ln|\cos y| = +xe^{-x} + e^{-x} + C$$

$$2\ln|\cos y| = xe^{-x} + e^{-x} - C$$ – general solution

condition $y(0) = 0$.

Plug in $x=0$ and $y=0$

$$2\ln|\cos 0| = 0e^{-0} + e^{-0} - C$$

$$0 = 1 - C \Rightarrow \boxed{C = 1}$$

$$\boxed{2\ln|\cos y| = xe^{-x} + e^{-x} - 1}$$ solution of the initial value problem.