seperable equations: $\frac{dy}{dx} = y' = f(x)g(y)$

$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

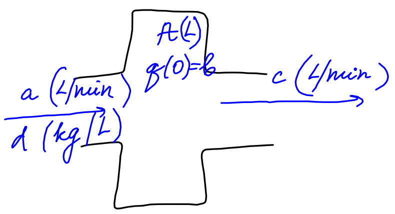
Linear equations: $y' + p(t)y = q(t)$

Integrating factor $\mu(t)$: $\frac{d\mu}{dt} = p(t)\mu$

$$\int \frac{d}{dt}[\mu(t)y(t)]dt = \int \mu(t)q(t)dt$$

Section 2.3 **Modeling with first order equations.**

1. **Mixing.** A brine solution of salt flows at a constant rate of a L/min into a large tank that initially held A L of brine solution in which was dissolved b kg of salt. The solution inside the tank is kept well stirred and flows out of the tank at the rate c L/min. If the concentration of salt in the brine entering the tank is d kg/L, determine the mass of salt in the tank after t min. $(c \neq a)$



$q(t)$ is the mass of salt in the tank @ time t

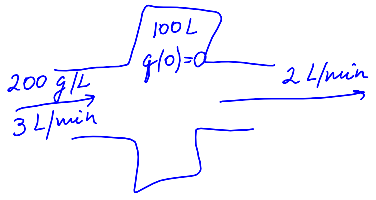
$$\frac{dq}{dt} = \boxed{\text{rate in}} - \boxed{\text{rate out}}$$

rate in $= ad$

rate out $= \dfrac{q(t)}{A+(a-c)t}\, c$

$$\boxed{\frac{dq}{dt} = ad - \frac{q(t)}{A+(a-c)t}\, c, \quad q(0) = b}$$

Example 1. A tank initially contains 100 L of fresh water. A brine containing 200 g/L of salt ~~salt~~ flows into the tank at rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the rate 2 L/min. Determine the concentration of salt at any time.



$q(t)$ is the mass of salt @ time t

$$\frac{dq}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = (200)(3) = 600$$

$$\text{rate out} = \frac{2q(t)}{100 + (3-2)t} = \frac{2q}{100+t}$$

Initial value problem:

$$\frac{dq}{dt} = 600 - \frac{2q}{100+t}, \quad q(0)=0$$

$$\frac{dq}{dt} + \frac{2}{100+t} q = 600 \Rightarrow p(t) = \frac{2}{100+t}$$

Integrating factor:

$$\frac{\frac{d\mu}{dt}\cancel{dt}}{\mu} = \frac{\frac{2}{100+t}\mu\, dt}{\mu}$$

$$\int \frac{d\mu}{\mu} = \int \frac{2}{100+t}\, dt$$

$$\ln|\mu| = 2\ln|100+t| + \cancel{C}^{0}$$

$$\mu(t) = (100+t)^2$$

$$\int \frac{d}{dt}\left[\mu(t) q(t)\right] dt = \int \mu(t)(600)\, dt$$

$$(100+t)^2 q(t) = \int 600(100+t)^2\, dt$$

$$\frac{(100+t)^2 q(t)}{(100+t)^2} = \frac{200(100+t)^3 + C}{(100+t)^2}$$

$$q(t) = 200(100+t) + \frac{C}{(100+t)^2} \quad \text{general solution}$$

$$0 = q(0) = 20000 + \frac{C}{10000}$$

$$C = -2\cdot 10^8$$

$$q(t) = 200(100+t) - \frac{2\times 10^8}{(100+t)^2} \quad \text{the mass of salt @ time } t.$$

concentration is $\boxed{\dfrac{q(t)}{100+t} = 200 - \dfrac{2\times 10^8}{(100+t)^3}}$

2. **Population models.** If we assume that the people die only of natural causes, we might expect the death rate also to be proportional to the size of the population. So,

$$\frac{dp}{dt} = k_1 p - k_2 p = (k_1 - k_2)p = kp,$$

where $k = k_1 - k_2$ and k_2 is the proportionality constant for the death rate. Let's assume that $k_1 > k_2$ so that $k > 0$. This gives the mathematical model

$$\boxed{\frac{dp}{dt} = kp, \qquad p(0) = p_0,}$$

which is called the **Malthusian** or **exponential, law** of population growth.

We might assume that another component of the death is proportional to the number of two-party interactions. There are $p(p-1)/2$ such possible interactions for a population of size p. Thus, if we combine the birth rate with the death rate and rearrange constants, we get the **logistic model**

$$\frac{dp}{dt} = -Ap(p - p_1), \qquad p(0) = p_0,$$

where $A = k_3/2$ and $p_1 = (2k_1/k_3) + 1$.

The function $p(t)$ is called the **logistic function**.

Example 2. The population of mosquitoes in a certain area increases at a rate proportional to the current population, and in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially and predators eat 20,000 mosquitoes/day. Determine the population of mosquitoes in the area at any time.

$p(t)$ is the population @ time t, t is in days.

no predators: $p(7) = 2p(0)$

$\frac{dp}{dt} = kp$, k is an unknown constant

$\int \frac{dp}{p} = \int k\,dt$

$\ln|p| = kt + C$

$p(t) = e^{kt+C} = \cancel{e^C} \cdot e^{kt}$ C_1

$p(t) = C_1 e^{kt}$

$p(0) = C_1$, $p(7) = C_1 e^{7k}$

$\cancel{C_1} e^{7k} = 2\cancel{C_1}$ solve for $\boxed{k = \frac{\ln 2}{7}}$

with predators: $\frac{dp}{dt} = kp - 20{,}000$, $p(0) = 200{,}000$

integrating factor: $\frac{d\mu}{dt} = -k\mu$

$\int \frac{d\mu}{\mu} = -\int k\,dt$

$\ln|\mu| = -kt$

$\mu(t) = e^{-kt}$

$\frac{d}{dt}\left[e^{-kt}p(t)\right] = e^{-kt}(-20{,}000)$

$e^{-kt}p(t) = \int e^{-kt}(-20{,}000)\,dt = \frac{20{,}000}{k}e^{-kt} + C$

$p(t) = \frac{20{,}000}{k} + Ce^{kt}$

$p(0) = \frac{20{,}000}{k} + C = 200{,}000$

$C = 200{,}000 - \frac{20{,}000}{k}$

$$\boxed{p(t) = \frac{20{,}000}{k} + \left(200{,}000 - \frac{20{,}000}{k}\right)e^{kt}, \quad k = \frac{\ln 2}{7}}$$

3. **Heating and cooling of buildings.** Let $T(t)$ represent the temperature inside the building at time t and view the building as a single compartment.

We will consider three main factors that affect the temperature inside the building. First is the heat produced by people, lights, and machines inside the building. This causes a rate of increase in temperature that we will denote by $H(t)$. Second is the heating (or cooling) supplied by the furnace (or air conditioner). This rate of increase (or decrease) in temperature will be represented by $U(t)$. The third factor is the effect of the outside temperature $M(t)$ on the temperature inside the building. Third factor can be modeled using **Newtons law of cooling**. This law states that

$$\boxed{\frac{dT}{dt} = K(M(t) - T(t)).}$$

The positive constant K depends on the physical properties of the building, K does not depend on M, T or t.

Summarizing, we have

$$\frac{dT}{dt} = K(M(t) - T(t)) + U(t) + H(t),$$

where $H(t) \geq 0$ and $U(t) > 0$ for furnace heating and $U(t) < 0$ for air conditioning cooling.

Example 3. An object with temperature 150^0 is placed in a freezer whose temperature is 30^0. Assume that the temperature of the freezer remains essentially constant. If the object is cooled to 120^0 after 8 min, what will its temperature be after 18 min?

$T(t)$ is the temperature of the object @ time t.

$M = 30°$ (outside temperature)

$T(0) = 150$, $T(8) = 120$, $T(18) = ?$

$$\frac{dT}{dt} = k(M - T)$$

$$\frac{dT}{dt} = k(30 - T),$$ k is an unknown constant.

$$\int \frac{dT}{30-T} = \int k\,dt$$

$$-\ln|30-T| = kt + C$$

$$\ln|30-T| = -kt - C$$

$$30 - T = e^{-kt-c} = \cancel{e^{-c}}^{\,c_1} \cdot e^{-kt}$$

$$T = 30 - c_1 e^{-kt}$$

$$\underline{150} = T(0) = \underline{30 - c_1}$$

$$c_1 = -120$$

$$\boxed{T(t) = 30 + 120 e^{-kt}}$$

$$120 = T(8) = 30 + 120 e^{-8k}$$

$$90 = 120 e^{-8k} \Rightarrow e^{-8k} = \frac{90}{120} = \frac{3}{4}$$

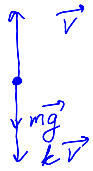
$$-8k = \ln\frac{3}{4}$$

$$\boxed{k = -\frac{1}{8}\ln\frac{3}{4}}$$

$$\boxed{T(18) = 30 + 120 e^{+\frac{1}{8}\ln\frac{3}{4}(18)}}$$

4. **Falling body.** (See Chapter 1)

Example 4. A body of mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Assume that the gravitational attraction of the earth is constant. Find the velocity of the body at any time.



net force $\vec{F} = m\vec{g} + k\vec{v}$

$$m\vec{a} = m\vec{g} + k\vec{v}$$

$$a = \frac{dv}{dt}$$

component form: $\dfrac{m\frac{dv}{dt}}{m} = \dfrac{-mg - kv}{m}$, $v(0) = v_0$

$$\frac{\frac{dv}{dt}dt}{g+\frac{k}{m}v} \quad \frac{\left(-g - \frac{k}{m}v\right)dt}{g+\frac{k}{m}v}$$

$$\int \frac{dv}{g+\frac{k}{m}v} = -\int dt$$

$$\frac{k}{m}\frac{m}{k}\ln\left|g+\frac{k}{m}v\right| = (-t+C)\frac{k}{m}$$

$$\ln\left|g+\frac{k}{m}v\right| = \frac{k}{m}(-t+c)$$

$$g+\frac{k}{m}v = e^{\frac{k}{m}(-t+c)} = e^{-t\frac{k}{m}} \cdot \overset{C_1}{\cancel{e^{c\frac{k}{m}}}}$$

$$g+\frac{k}{m}v = C_1 e^{-t\frac{k}{m}}$$

$$\frac{k}{m}v = C_1 e^{-t\frac{k}{m}} - g$$

$$v = C_1\frac{m}{k}e^{-t\frac{k}{m}} - \frac{gm}{k}$$

$$v_0 = v(0) = \frac{C_1 m}{k} - \frac{gm}{k}$$

$$C_1\frac{m}{k} = v_0 + \frac{gm}{k}$$

$$C_1 = v_0\frac{k}{m} + g$$

$$v(t) = \left(v_0\frac{k}{m} + g\right)\frac{m}{k}e^{-\frac{k}{m}t} - \frac{gm}{k}$$

$$\boxed{v(t) = \left(v_0 + \frac{gm}{k}\right)e^{-\frac{k}{m}t} - \frac{gm}{k}}$$