Section 2.4 **Differences between linear and nonlinear equations.**

- **Linear equations.**

Theorem 1. Suppose $p(t)$ and $q(t)$ are continuous on some interval I that contains the point t_0. Then for any choice of initial value y_0, there exists a unique solution $y(t)$ on I to the initial value problem

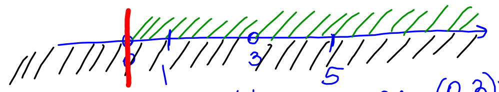
$$y' + p(t)y = q(t), \qquad y(t_0) = y_0$$

Example 1. Determine (without solving the problem) an interval in which the solution of the given IVP is certain to exist:

$$\frac{(t-3)y' + (\ln t)y}{t-3} = \frac{2t}{t-3} \Rightarrow y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3}$$

$p(t) = \frac{\ln t}{t-3}$ - continuous on $(0,3)\cup(3,\infty)$

$q(t) = \frac{2t}{t-3}$ continuous on $(-\infty,3)\cup(3,\infty)$



both are continuous on $(0,3)\cup(3,\infty)$

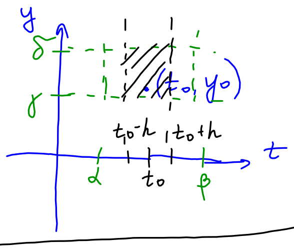
1. $y(1) = 2$
   $\boxed{(0,3)}$

2. $y(5) = 6$
   $\boxed{(3,\infty)}$

- Nonlinear equations:

Theorem 2. Let the functions f and $\dfrac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained $\alpha < t < \beta$, there is a unique solution of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$



Note: Thm 1 follows from Thm 2.

For a linear eqn.

$$y' + p(t)y = g(t) \Rightarrow y' = g(t) - p(t)y$$

thus, $f(t,y) = g(t) - p(t)y$

$$\frac{\partial f}{\partial y} = -p(t)$$

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Remarks:

1. By this theorem we can guarantee the existence of solution only for values of t which are suffiently closed to t_0, but not for all t.
2. Geometric consequence of the theorem is that two integral curves never intersect each other.
3. The condition "$\dfrac{\partial f}{\partial y}$ is continuous in some rectangle..." is important for uniqueness.

Example 2. Solve the initial value problem

$$y' = y^{1/3}, \; y(0) = 0$$

Try $\boxed{y = 0} \Rightarrow y' = 0$, and $y(0) = 0$.

$\dfrac{dy}{dt} = y^{1/3}$ – separable

$$\int \frac{dy}{y^{1/3}} = \int dt$$

$$\frac{y^{-1/3+1}}{-1/3+1} = t + C \Rightarrow \frac{3}{2} y^{2/3} = t + C$$

$$y^{2/3} = \frac{2}{3}(t+C)$$

plug in $y = t = 0$, we'll get $0 = \frac{2}{3}(0+C)$ or $C = 0$

$$\boxed{y^{2/3} = \frac{2}{3}t}$$

$f(t,y) = y^{1/3}$ – continuous @ $y = 0$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{3} y^{-2/3} = \dfrac{1}{3y^{2/3}}$ discontinuous @ $y = 0$

Example 3. For the IVP

$$y' = \frac{\ln|ty|}{1 - t^2 + y^2} \quad , \quad y(t_0) = y_0$$

state where in the ty-plane the hypotheses of Theorem 2 are satisfied.

$$f(t,y) = \frac{\ln|ty|}{1-t^2+y^2} \qquad |ty| > 0 \text{ whenever } ty \neq 0$$

$$1 - t^2 + y^2 \neq 0$$

$$-1 \neq y^2 - t^2 \quad \text{or} \quad t^2 - y^2 \neq 1$$

$$\frac{\partial f}{\partial y} = \frac{\frac{1}{y}(1-t^2+y^2) - 2y \ln|ty|}{(1-t^2+y^2)^2} = \frac{1-t^2+y^2 - 2y^2 \ln|ty|}{y(1-t^2+y^2)}$$

$$ty \neq 0, \; y \neq 0, \; 1 - t^2 + y^2 \neq 0$$