Section 2.5 **Autonomous equations.**

**Definition.** A first order differential equation of the form

$$\frac{dy}{dt} = f(y)$$

is called **autonomous** (the independent variable does not appear explicitly).

**Definition.** The constant solutions of the equation $\frac{dy}{dt} = f(y)$ are called **equilibrium solutions**. They correspond to the zeros of $f$. Zeros of $f$ are called **critical points**.

**Definition.** An equilibrium solution is **stable** if any neighboring solution is attracted to it as t goes to infinity. An equilibrium solution is **unstable** if any neighboring solution is repelled by the solution. An equilibrium solution that is not stable or unstable is called **semistable**.

Let $y = y_0$ is an equilibrium solution. Then

- $y = y_0$ is **stable** if $f(y)$ changes sign at $y_0$ from "+" to "-".
- $y = y_0$ is **unstable** if $f(y)$ changes sign at $y_0$ from "-" to "+".
- $y = y_0$ is **semistable** if $f(y)$ does not change sign at $y_0$.

For autonomous systems the slope on horizontal lines $y = y_0$ is the same and the qualitative analysis can be made on the so called **phase line portrait**.

**Directions to draw phase line portrait:**

1. Find all equilibrium points (i.e. roots of $f(y) = 0$), draw a horizontal line and mark those points on it.
2. Check the sign of $f(y)$ in each of the intervals determined by the equilibrium points. Over those intervals where $f(y) > 0$ draw arrows pointing to the right, and on those where $f(y) < 0$ draw arrows pointing to the left (indicating in which direction are solutions owing).

**Example 1.** For the equation $\frac{dy}{dt} = y^2 - 1$

1. Sketch the graph of $f$ versus $y$.

2. Find all the equilibrium solutions of the equation.
3. Sketch the direction field of the equation.
4. Let $y(t)$ be the solution of the equation satisfying the initial condition $y(0) = y_0$, where $-\infty \leq y_0 \leq \infty$. Find the limit of $y(t)$ when $t \to \pm\infty$.

5. Carry out a phase line analysis for the equation.

6. Solve the equation.