Section 2.5 **Autonomous equations.**

Definition. A first order differential equation of the form

$$\boxed{\frac{dy}{dt} = f(y)}$$

is called autonomous (the independent variable does not appear explicitly).

Definition. The constant solutions of the equation $\frac{dy}{dt} = f(y)$ are called **equilibrium solutions**. They correspond to the zeros of f. Zeros of f are called **critical points**.

Definition. An equilibrium solution is **stable** if any neighboring solution is attracted to it as t goes to infinity. An equilibrium solution is **unstable** if any neighboring solution is repelled by the solution. An equilibrium solution that is not stable or unstable is called **semistable**.

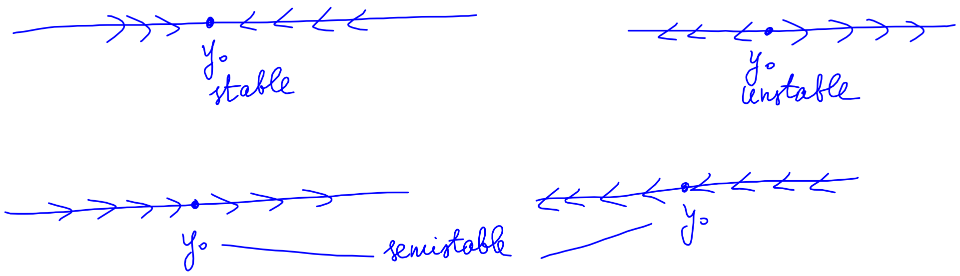
Let $y = y_0$ is an equilibrium solution. Then

- $y = y_0$ is **stable** if $f(y)$ changes sign at y_0 from "+" to "-".
- $y = y_0$ is **unstable** if $f(y)$ changes sign at y_0 from "-" to "+".
- $y = y_0$ is **semistable** if $f(y)$ does not change sign at y_0.

For autonomous systems the slope on horizontal lines $y = y_0$ is the same and the qualitative analysis can be made on the so called **phase line portrait**.

Directions to draw phase line portrait:

1. Find all equilibrium points (i.e. roots of $f(y) = 0$), draw a horizontal line and mark those points on it.
2. Check the sign of $f(y)$ in each of the intervals determined by the equilibrium points. Over those intervals where $f(y) > 0$ draw arrows pointing to the right, and on those where $f(y) < 0$ draw arrows pointing to the left (indicating in which direction are solutions owing).



Example 1. For the equation $\frac{dy}{dt} = y^2 - 1$

[redacted]

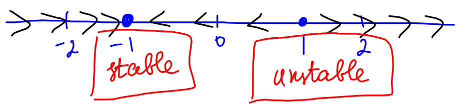
1. Find all equilibrium solutions of the equation.

$y^2 - 1 = 0 \Rightarrow y = \pm 1$

$y = 1,\ y = -1$ equilibrium solutions.

2. Carry out the phase line analysis for the equation.

$f(y) = y^2 - 1$



$f(0) = -1 < 0$

$f(2) = 4 - 1 > 0$

$f(-2) = 4 - 1 > 0$

$y = -1$ stable

$y = 1$ unstable

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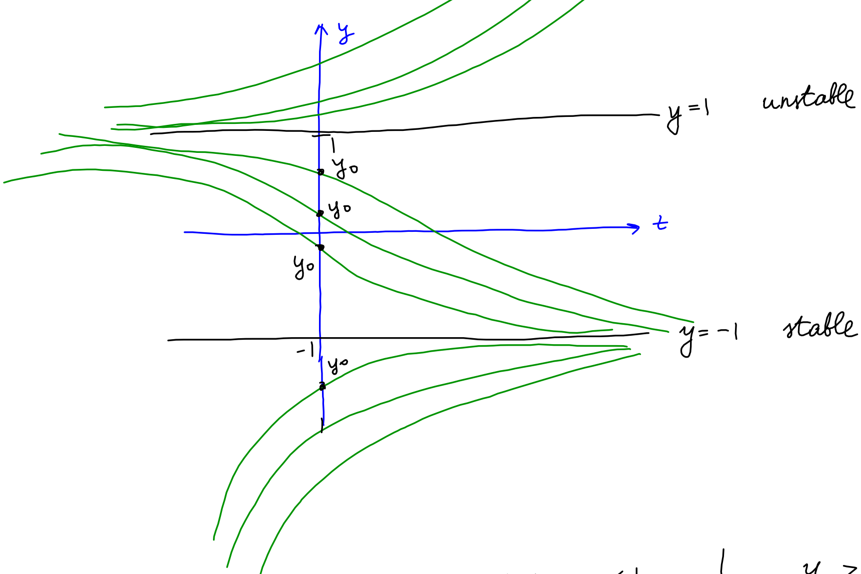
2. Find all the equilibrium solutions of the equation.

3. Sketch the direction field of the equation.

~~4~~ 3. Let $y(t)$ be the solution of the equation satisfying the initial condition $y(0) = y_0$, where $-\infty \leq y_0 \leq \infty$. Find the limit of $y(t)$ when $t \to \pm\infty$.

$y(0) = y_0$



$y_0 < -1$	$-1 < y_0 < 1$	$y_0 > 1$
$\lim_{t\to\infty} y(t) = -1$	$\lim_{t\to\infty} y(t) = -1$	$\lim_{t\to\infty} y(t) = \infty$
$\lim_{t\to-\infty} y(t) = -\infty$	$\lim_{t\to-\infty} y(t) = 1$	$\lim_{t\to-\infty} y(t) = 1$

[redacted]

4. Solve the equation. $\frac{dy}{dt} = y^2 - 1$ separable

$$\int \frac{dy}{y^2-1} = \int dt$$

Partial fractions: $\frac{1}{y^2-1} = \frac{A}{y-1} + \frac{B}{y+1}$

$$1 = A(y+1) + B(y-1)$$

$y=-1$: $1 = -2B \Rightarrow B = -1/2$

$y=1$: $1 = 2A \Rightarrow A = 1/2$

$$\frac{1}{y^2-1} = \frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right)$$

$$\frac{1}{2}\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy = \int dt$$

$$\boxed{\frac{1}{2}\left[\ln|y-1| - \ln|y+1|\right] = t + C}$$