Section 2.6 **Exact equations and integrating factors.**

Given an equation

$$M(x,y)dx + N(x,y)dy = 0. \quad (*)$$

There exists an implicit solution of the equation $F(x,y) = C$ if and only if

if $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ then $(*)$ is exact

$\boxed{F(x,y)=C}$ such that

$F(x,y)$ satisfies the following conditions:

$$\frac{\partial F(x,y)}{\partial x} = M(x,y), \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

, is the general solution of $(*)$.

In such a case, the equation is called **exact**.

Example 1. Is the equation

$$\underbrace{(3x^2 - 2xy + 2)}_{M(x,y)}dx + \underbrace{(6y^2 - x^2 + 3)}_{N(x,y)}dy = 0$$

exact? If it is, solve it.

$\frac{\partial M}{\partial y} = -2x$

$\frac{\partial N}{\partial x} = -2x$

match, thus the equation is exact

Find $F(x,y)$ such that

$$\begin{cases} \int \frac{\partial F}{\partial x} dx = M(x,y) = \int [3x^2 - 2xy + 2] dx \Rightarrow F(x,y) = x^3 - x^2 y + 2x + g(y) \\ \frac{\partial F}{\partial y} = N(x,y) = 6y^2 - x^2 + 3 \end{cases}$$

$g(y)$ is an unknown function

$$\frac{\partial F}{\partial y} = 0 - \cancel{x^2} + 0 + g'(y) = 6y^2 - \cancel{x^2} + 3 \Rightarrow \int g'(y) dy = \int [6y^2 + 3] dy$$

$$g(y) = 2y^3 + 3y + C$$

$$F(x,y) = x^3 - x^2 y + 2x + 2y^3 + 3y + C$$

General solution: $\boxed{x^3 - x^2 y + 2x + 2y^3 + 3y + C = 0}$

Example 2. Solve IVP:

$$3x^2 - y + (2y - x)y' = 0, \qquad y(1) = 3$$

$$y' = \frac{dy}{dx}$$

$$dx\left((3x^2 - y) + (2y - x)\frac{dy}{dx}\right) = 0\,(dx)$$

$$\underbrace{(3x^2 - y)}_{M(x,y)}\,dx + \underbrace{(2y - x)}_{N(x,y)}\,dy = 0$$

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x} = -1 \qquad \text{exact}$$

$$F(x,y): \begin{cases} \dfrac{\partial F}{\partial x} = 3x^2 - y \\ \displaystyle\int \frac{\partial F}{\partial y}\,dy = \int (2y - x)\,dy \end{cases} \Rightarrow \quad F(x,y) = y^2 - xy + g(x)$$

$g(x)$ is an unknown function.

$$\frac{\partial F}{\partial x} = -y + g'(x) = 3x^2 - y$$

$$g'(x) = 3x^2 \Rightarrow g(x) = x^3 + C$$

$$F(x,y) = y^2 - xy + x^3 + C$$

General solution is $y^2 - xy + x^3 + C = 0$

Plug in $y = 3$ and $x = 1$: $9 - 3 + 1 + C = 0 \Rightarrow C = -7$

$\boxed{y^2 - xy + x^3 - 7 = 0}$ solution of the initial value problem.

Example 3. Is the equation

$y' = \frac{dy}{dx}$

$$x^2y^3 + x(1+y^2)y' = 0$$

exact?

$x^2y^3 + x(1+y^2)\frac{dy}{dx} = 0$ or $\underbrace{x^2y^3}_{M(x,y)}dx + \underbrace{x(1+y^2)}_{N(x,y)}dy = 0$

$\frac{\partial M}{\partial y} = 3x^2y^2 \neq \frac{\partial N}{\partial x} = 1+y^2$

$\boxed{\text{it is no exact}}$

Multiply the equation by the integrating factor $\mu(x, y) = \dfrac{1}{xy^3}$ and then solve it.

$\frac{1}{xy^3}\left[x^2y^3\,dx + x(1+y^2)\,dy\right] = 0$

$x\,dx + \frac{1+y^2}{y^3}\,dy = 0.$

$\underbrace{x}_{M(x,y)}dx + \underbrace{\left(\frac{1}{y^3} + \frac{1}{y}\right)}_{N(x,y)}dy = 0.$

$\frac{\partial M}{\partial y} = 0 \quad , \quad \frac{\partial N}{\partial x} = 0$

Find $F(x,y)$ such that

$\begin{cases} \frac{\partial F}{\partial x} = x \\ \int \frac{\partial F}{\partial y}dy = \int \left[\frac{1}{y^3} + \frac{1}{y}\right]dy \end{cases}$

$F(x,y) = \frac{y^{-2}}{-2} + \ln|y| + g(x)$

$g(x)$ is an unknown function.

$\frac{\partial F}{\partial x} = g'(x) = x \Rightarrow g(x) = \frac{x^2}{2} + C$

$F(x,y) = -\frac{1}{2y^2} + \ln|y| + \frac{x^2}{2} + C$

General solution $\boxed{-\frac{1}{2y^2} + \ln|y| + \frac{x^2}{2} + C = 0}$

Given an equation

$$M(x,y)dx + N(x,y)dy = 0.$$

If $\dfrac{M_y - N_x}{N(x,y)}$ is a function of x only, then a solution $\mu(x)$ of the equation

$$\boxed{\frac{d\mu}{dx} = \frac{M_y - N_x}{N(x,y)}\mu}$$

is an integrating factor for the differential equation.

Example 4. Find an integrating factor for the equation

$$\underbrace{(3x^2y + 2xy + y^3)}_{M(x,y)}dx + \underbrace{(x^2 + y^2)}_{N(x,y)}dy = 0$$

and then solve the equation.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N(x,y)} = \frac{3x^2 + \cancel{2x} + 3y^2 - (\cancel{2x})}{x^2+y^2} = \frac{3(x^2+y^2)}{x^2+y^2} = 3$$

Integrating factor $\mu(x)$: $\dfrac{d\mu}{dx} = 3\mu$

$$\frac{d\mu}{\mu} = 3dx \Rightarrow \ln|\mu| = 3x + \cancel{C}^{\,0}$$

$$\mu = e^{3x}$$

$$e^{3x}\left[(3x^2y+2xy+y^3)\,dx + (x^2+y^2)\,dy\right] = 0$$

$$\underbrace{e^{3x}(3x^2y+2xy+y^3)}_{M(x,y)}dx + \underbrace{e^{3x}(x^2+y^2)}_{N(x,y)}dy = 0.$$

$$\frac{\partial M}{\partial y} = e^{3x}(3x^2+2x+3y^2)$$

$$\frac{\partial N}{\partial x} = 3e^{3x}(x^2+y^2) + e^{3x}(2x)$$

match, $\boxed{\text{exact}}$

Find $F(x,y)$ such that

$$\begin{cases} \dfrac{\partial F}{\partial x} = e^{3x}(3x^2y+2xy+y^3) \\ \displaystyle\int \frac{\partial F}{\partial y}dy = \int e^{3x}(x^2+y^2)\,dy \end{cases} \Rightarrow F(x,y) = e^{3x}\left(x^2y + \frac{y^3}{3}\right) + g(x)$$

$$\frac{\partial F}{\partial x} = 3e^{3x}\left(\cancel{x^2y} + \cancel{\frac{y^3}{3}}\right) + e^{3x}(\cancel{2xy}) + g'(x) = e^{3x}(\cancel{3x^2y} + \cancel{2xy} + \cancel{y^3})$$

$$g'(x) = 0, \quad g(x) = C$$

$$F(x,y) = e^{3x}\left(x^2y + \frac{y^3}{3}\right) + C$$

General solution $\boxed{e^{3x}\left(x^2y + \frac{y^3}{3}\right) + C = 0}$