Section 3.1 **Homogeneous Equations with Constant Coefficients**

We begin our discussion with homogeneous equations with constant coefficients

$$\boxed{ay'' + by' + cy = 0}$$

where a, b, c are constants.

We try to find a solution of the form $y = e^{rt}$. , $y' = re^{rt}$, $y'' = r^2e^{rt}$

$$a\underbrace{(r^2e^{rt})}_{y''} + b\underbrace{(re^{rt})}_{y'} + \underbrace{ce^{rt}}_{y} = 0$$

$$e^{rt}\left[ar^2 + br + c\right] = 0$$

or

$$ar^2 + br + c = 0$$

Definition. An equation

$$\boxed{ar^2 + br + c = 0}$$

is called the auxiliary equation or characteristic equation associated with equation $ay'' + by' + cy = 0$.

If $b^2 - 4ac > 0$, then the auxiliary equation has two distinct real roots r_1 and r_2. Then

$$\boxed{y(x) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

is the general solution to the equation.

$y_1(t) = e^{r_1 t}$

$y_2(t) = e^{r_2 t}$, $r_1 \neq r_2$

$$W[y_1, y_2] = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix}$$

$$= r_2 e^{r_1 t} e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t}$$

$$= e^{(r_1 + r_2)t} (r_2 - r_1) \neq 0$$

$\{e^{r_1 t}, e^{r_2 t}\}$ form the fundamental solution set.

Example 1. Find the general solution to the given equation

(a) $y'' - y' - 2y = 0$.

auxiliary equation: $y'' \to r^2$, $y' \to r$, $y \to 1$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \Rightarrow r_1 = 2, \quad r_2 = -1$

$$\boxed{y(t) = c_1 e^{2t} + c_2 e^{-t}}$$

(b) $y'' + 7y' + 10y = 0$.

auxiliary equation

$r^2 + 7r + 10 = 0$

$(r+2)(r+5) = 0$

$r_1 = -2, \quad r_2 = -5$

$$\boxed{y(t) = c_1 e^{-2t} + c_2 e^{-5t}}$$

Example 2. Solve the given initial value problems.

(a) $y'' + y' = 0$, $y(0) = 2$, $y'(0) = 1$.

auxiliary equation

$y'' \to r^2$
$y' \to r$
$y \to 1$

$r^2 + r = 0$ or $r(r+1) = 0$

$r_1 = 0$, $r_2 = -1$

general solution is $y(t) = C_1 e^{0 \cdot t} + C_2 e^{-t}$

$y(t) = C_1 + C_2 e^{-t}$

$y'(t) = -C_2 e^{-t}$

$y(0) = C_1 + C_2 = 2$

$y'(0) = -C_2 = 1$

$C_2 = -1$, $C_1 = 2 - C_2 = 3$

$$\boxed{y(t) = 3 - e^{-t}}$$

(b) $y'' + 4y' - 5y = 0$, $y(0) = 11$, $y'(0) = -7$.

auxiliary equation

$$r^2 + 4r - 5 = 0$$

$$(r+5)(r-1) = 0$$

$$r_1 = -5, \quad r_2 = 1$$

General solution

$$y(t) = C_1 e^{-5t} + C_2 e^{t} \qquad y(0) = C_1 + C_2 = 11$$

$$y'(t) = -5C_1 e^{-5t} + C_2 e^{t} \qquad y'(0) = -5C_1 + C_2 = -7$$

$$\begin{cases} C_1 + C_2 = 11 \Rightarrow & C_2 = 11 - C_1 \\ -5C_1 + C_2 = -7 \end{cases}$$

$$-5C_1 + (11 - C_1) = -7$$

$$-6C_1 = -18 \rightarrow C_1 = 3, \quad C_2 = 8$$

$$\boxed{y(t) = 3e^{-5t} + 8e^{t}}$$