Section 3.2 **Solutions of linear homogeneous equations; the Wronskian.**

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

where f is some given function.

An initial value problem consists of a differential equation together with the pair of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

A second order ordinary differential equation is said to be **linear** if it is written in the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

or

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t). \tag{1}$$

If $g(t) = 0$, then the equation

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

is called **homogeneous**. Otherwise, the equation is called **nonhomogeneous**.

Theorem 1 (existence and uniqueness of solution). Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on some interval (a, b) that contains the point t_0. Then, for any choice of initial values y_0, y_1 there exists a unique solution $y(t)$ on the whole interval (a, b) to the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(0) = y_1.$$

Example 1. Find the largest interval for which Theorem 2 ensures the existence and uniqueness of solution to the initial value problem

$$e^t y'' - \frac{y'}{t-3} + y = \ln t,$$

$$y(1) = y_0, \qquad y'(1) = y_1,$$

where y_0 and y_1 are real constants.

Theorem 2 (Principle of superposition). Let y_1 and y_2 be solutions to the *homogeneous* equation (2). Then any linear combination $C_1y_1 + C_2y_2$ of y_1 and y_2, where C_1 and C_2 are constants, is also the solution to (2).

Example 2. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $y = c_1 + c_2t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 1.

Definition For any two differentiable functions y_1 and y_2, the determinant

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

is called the **Wronskian** of y_1 and y_2.

Example 3. Find the Wronskian for the functions $e^t \sin t$, $e^t \cos t$.

Example 4. If the Wronskian of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

Definition 2. A pair of solutions $\{y_1, y_2\}$ to $y'' + p(t)y' + q(t)y = 0$ on I is called **fundamental solution set** if

$$W[y_1, y_2](t_0) \neq 0$$

at some $t_0 \in I$.

Theorem 3. (Fundamental solutions of homogeneous equations) Let y_1 and y_2 denote two solutions on I to

$$y'' + p(t)y' + q(t)y = 0,$$

where $p(t)$ and $q(t)$ are continuous on I. Suppose at some point $t_0 \in I$ these solutions satisfy

$$W[y_1, y_2](t_0) \neq 0. \tag{3}$$

Then every solution to (2) on I can be expressed in the form

$$y(t) = C_1 y_1(t) + C_2 y_2(t), \tag{4}$$

where C_1 and C_2 are constants.

Theorem 4. (Abels Theorem) If y_1 and y_2 are solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I, then the Wronskian is given by

$$W(y_1, y_2)(t) = c \exp\left[\int p(t)dt\right],$$

where c is a certain constant that depends on y_1 and y_2, but not on t.