Section 3.2 **Solutions of linear homogeneous equations; the Wronskian.**

A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

where f is some given function.

An initial value problem consists of a differential equation together with the pair of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

A second order ordinary differential equation is said to be **linear** if it is written in the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

or

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t). \tag{1}$$

If $g(t) = 0$, then the equation

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

is called **homogeneous**. Otherwise, the equation is called **nonhomogeneous**.

Theorem 1 (existence and uniqueness of solution). Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on some interval (a, b) that contains the point t_0. Then, for any choice of initial values y_0, y_1 there exists a unique solution $y(t)$ on the whole interval (a, b) to the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y_1.$$

Example 1. Find the largest interval for which Theorem 1 ensures the existence and uniqueness of solution to the initial value problem

$$e^t y'' - \frac{y'}{t-3} + y = \ln t, \quad \Rightarrow \quad y'' - \frac{1}{e^t(t-3)} y' + \frac{1}{e^t} y = \frac{\ln t}{e^t}$$

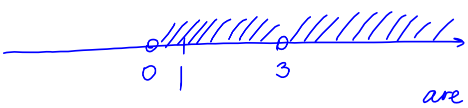
$$y(1) = y_0, \quad y'(1) = y_1,$$

where y_0 and y_1 are real constants.

$p(t) = -\frac{1}{e^t(t-3)}$ continuous if $t \neq 3$

$q(t) = \frac{1}{e^t} = e^{-t}$ continuous for all t

$g(t) = \frac{\ln t}{e^t}$ continuous for $t > 0$



all three functions are continuous on $(0,3) \cup (3, \infty)$

$\boxed{(0,3)}$

$y'' + p(t)y' + q(t)y = 0 \qquad (2)$

Theorem 2 (Principle of superposition). Let y_1 and y_2 be solutions to the *homogeneous* equation (2). Then any linear combination $C_1y_1 + C_2y_2$ of y_1 and y_2, where C_1 and C_2 are constants, is also the solution to (2).

Example 2. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $y = c_1 + c_2t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 1.

$yy'' + (y')^2 = 0$

$y = C_1 + C_2\sqrt{t}$

$y' = \frac{C_2}{2\sqrt{t}}, \quad y'' = C_2 \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) t^{-3/2} = -\frac{C_2}{4t^{3/2}}$

$yy'' + (y')^2 = -\frac{C_2}{4t^{3/2}} \cdot \left(C_1 + C_2\sqrt{t}\right) + \frac{C_2^2}{4t} \neq 0$

Definition For any two differentiable functions y_1 and y_2, the determinant

$$\boxed{W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)}$$

is called the Wronskian of y_1 and y_2.

Example 3. Find the Wronskian for the functions $e^t \sin t$, $e^t \cos t$.

$$W = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \sin t + e^t \cos t & e^t \cos t - e^t \sin t \end{vmatrix}$$

$$= e^t \sin t\left(e^t \cos t - e^t \sin t\right) - e^t \cos t\left(e^t \sin t + e^t \cos t\right)$$

$$= e^{2t}\left[\cancel{\sin t \cos t} \underbrace{- \sin^2 t - \cos^2 t}_{-1} - \cancel{\sin t \cos t}\right] = -e^{2t} \neq 0 \text{ for all } t.$$

Example 4. If the Wronskian of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

$$W[f,g] = \begin{vmatrix} e^{2t} & g(t) \\ 2e^{2t} & g'(t) \end{vmatrix} = \frac{e^{2t} g'(t) - 2e^{2t} g(t)}{e^{2t}} = \frac{3e^{4t}}{e^{2t}}$$

$$g'(t) - 2g(t) = 3e^{2t}$$ linear 1st order

Integrating factor $\mu(t)$: $\frac{d\mu}{dt} = -2\mu$

$$\frac{d\mu}{\mu} = -2dt \Rightarrow \mu(t) = e^{-2t}$$

$$g(t)e^{-2t} = \int 3e^{2t} e^{-2t} dt = 3t + C$$

$$\boxed{g(t) = 3te^{2t} + Ce^{+2t}}$$

Definition 2. A pair of solutions $\{y_1, y_2\}$ to $y'' + p(t)y' + q(t)y = 0$ on I is called **fundamental solution set** if

$$W[y_1, y_2](t_0) \neq 0$$

at some $t_0 \in I$.

Theorem 3. **(Fundamental solutions of homogeneous equations)** Let y_1 and y_2 denote two solutions on I to

$$y'' + p(t)y' + q(t)y = 0,$$

where $p(t)$ and $q(t)$ are continuous on I. Suppose at some point $t_0 \in I$ these solutions satisfy

$$W[y_1, y_2](t_0) \neq 0. \tag{3}$$

Then every solution to (2) on I can be expressed in the form

$$\boxed{y(t) = C_1 y_1(t) + C_2 y_2(t),} \tag{4}$$

where C_1 and C_2 are constants.

Theorem 4. **(Abels Theorem)** If y_1 and y_2 are solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I, then the Wronskian is given by

$$\boxed{W(y_1, y_2)(t) = c \exp\left[\int p(t)dt\right],}$$

where c is a certain constant that depends on y_1 and y_2, but not on t.