Section 3.3 **Complex roots of the characteristic equation.**

Complex numbers review.

A **complex number** is represented in the form $z = x + iy$, where x and y are real numbers satisfying the usual rules of addition and multiplication, and the symbol i, called the **imaginary unit**, has the property

$$\boxed{i^2 = -1.}$$

The numbers x and y are called the **real** and **imaginary** part of z, respectively, and are denoted by

$$x = \Re(z), y = \Im(z).$$

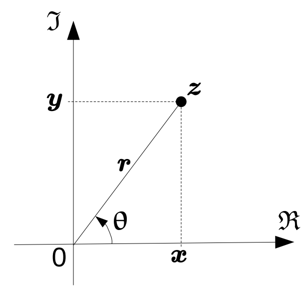
We say that z is real if $y = 0$, while it is purely imaginary if $x = 0$.

If $z = x + iy$, then a number $\bar{z} = x - iy$ is called the **complex conjugate to** z.

Let $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

1. $z\bar{z} = x^2 + y^2$
2. $cz = cx + i(cy)$ for any constant c
3. $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$
4. $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$
5. $\dfrac{z_1}{z_2} = \dfrac{z_1\overline{z_2}}{z_2\overline{z_2}}$

Geometrically, complex numbers can be represented as points in the plane.



We will call the xy-plane, when viewed in this manner, the complex plane, with the x-axis designated as the real axis, and the y-axis as the imaginary axis. We designate the complex number zero as the origin. Thus, $x + iy = 0$ means $x = y = 0$.

The complex number $z = x + iy$ can be represented in

- **polar form** as $z = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.
- **exponential form** as $z = re^{i\theta}$.

Euler's formula: $\boxed{e^{a+ib} = e^a(\cos b + i\sin b)}$

Complex roots of the characteristic equation.

$$ay'' + by' + cy = 0$$

where a, b, c are constants and $b^2 - 4ac < 0$.

Then the auxiliary equation

$$ar^2 + br + c = 0$$

has two complex conjugate roots:

$$r_1 = -\frac{b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a}, \qquad r_2 = -\frac{b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a} = \overline{r_1}.$$

$$\Re(r_1) = -\frac{b}{2a} = \alpha$$

$$\Im(r_1) = \frac{\sqrt{4ac - b^2}}{2a} = \beta$$

Then the solution of the equation is

$$y(t) = e^{r_1 t} = e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

Then the fundamental solution set is

$$y_1(t) = \Re[y(t)] = e^{\alpha t}\cos\beta t, \qquad y_2(t) = \Im[y(t)] = e^{\alpha t}\sin\beta t$$

Let us find $W[y_1, y_2](t)$

If the auxiliary equation

$$ar^2 + br + cr = 0$$

has two complex-conjugate roots

$$r_{1,2} = -\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a},$$

then the general solution of the equation

$$ay'' + by' + cy = 0$$

is

$$y(t) = e^{\alpha t}\left(C_1\cos\beta t + C_2\sin\beta t\right),$$

where $\alpha = -\dfrac{b}{2a}$, $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$, C_1 and C_2 are arbitrary constants.

Example 1. Find the general solution of the equation

$$y'' - 2y' + 2y = 0$$

Example 2. Solve the initial value problem

$$y'' + 4y' + 5y = 0, \qquad y(0) = 1, y'(0) = 0.$$