Section 3.3 **Complex roots of the characteristic equation.**

Complex numbers review.

A **complex number** is represented in the form $\boxed{z = x + iy,}$ where x and y are real numbers satisfying the usual rules of addition and multiplication, and the symbol i, called the **imaginary unit**, has the property

$$\boxed{i^2 = -1.}$$

The numbers x and y are called the **real** and **imaginary** part of z, respectively, and are denoted by

$x = Re(z)$, $y = Im(z)$

$$x = \Re(z), y = \Im(z).$$

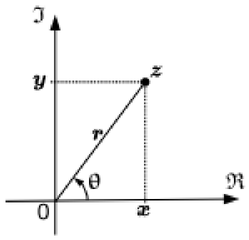
We say that z is real if $y = 0$, while it is purely imaginary if $x = 0$.

If $z = x + iy$, then a number $\bar{z} = x - iy$ is called the **complex conjugate to** z.

Let $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

1. $z\bar{z} = x^2 + y^2$
2. $cz = cx + i(cy)$ for any constant c
3. $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$
4. $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$
5. $\dfrac{z_1}{z_2} = \dfrac{z_1 \overline{z_2}}{z_2 \overline{z_2}}$

Geometrically, complex numbers can be represented as points in the plane.



We will call the xy-plane, when viewed in this manner, the complex plane, with the x-axis designated as the real axis, and the y-axis as the imaginary axis. We designate the complex number zero as the origin. Thus, $x + iy = 0$ means $x = y = 0$.

The complex number $z = x + iy$ can be represented in

- **polar form** as $z = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.
- **exponential form** as $z = re^{i\theta}$.

Euler's formula: $\boxed{e^{a+ib} = e^a(\cos b + i\sin b)}$

Example.

$z = 3+5i$

real part $\quad Re(z)=3$

imaginary part $\quad Im(z)=5$

$\bar{z} = 3-5i$ $\quad$ complex conjugate.

$z_1 = -4+i$

$Re(z_1) = -4$

$Im(z_1) = 1$

$\bar{z}_1 = -4-i$

$$\begin{aligned} z+z_1 &= (3+5i)+(-4+i) \\ &= (3-4)+i(5+1) \\ &= -1+6i \end{aligned}$$

$$\begin{aligned} z-2z_1 &= 3+5i-2(-4+i) \\ &= (3+8)+i(5-2) \\ &= 11+3i \end{aligned}$$

$$\begin{aligned} z\cdot z_1 &= (3+5i)(-4+i) \\ &= -12+3i-20i+5\cancel{i^2}^{-1} \\ &= -12-17i-5 \\ &= -17-17i \end{aligned}$$

$$\frac{z}{z_1} = \frac{3+5i}{-4+i} = \frac{(3+5i)(-4-i)}{(-4+i)(-4-i)} = \frac{-12-3i-20i-5\cancel{i^2}^{-1}}{(-4)^2-\cancel{(i)^2}^{-1}} = \frac{-12+5-23i}{17} = \frac{-7-23i}{17}$$

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absolute value of z

$z = 3+5i$

$$|z| = \sqrt{3^2+5^2} = \sqrt{9+25} = \sqrt{34}$$

$$\boxed{z = \sqrt{34}\left(\frac{3}{\sqrt{34}} + i\,\frac{5}{\sqrt{34}}\right) = r(\cos\theta + i\sin\theta)}$$ polar form

$$r = \sqrt{34}, \quad \cos\theta = \frac{3}{\sqrt{34}}, \quad \sin\theta = \frac{5}{\sqrt{34}}$$

$$\tan\theta = \frac{5}{3}, \quad \theta = \arctan\left(\frac{5}{3}\right)$$

exponential form $\quad z = 3+5i = \sqrt{34}\,e^{i\theta} = \sqrt{34}\,e^{i\arctan\left(\frac{5}{3}\right)}$

Euler's formula

$$e^{3+5i} = e^3\cdot e^{5i} = e^3(\cos 5 + i\sin 5)$$

$$e^{-4+i} = e^{-4}\cdot e^{i} = e^{-4}(\cos 1 + i\sin 1)$$

Complex roots of the characteristic equation.

$$ay'' + by' + cy = 0$$

where a, b, c are constants and $b^2 - 4ac < 0$.

Then the auxiliary equation

$$ar^2 + br + c = 0$$

has two complex conjugate roots:

$$r_1 = -\frac{b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a}, \qquad r_2 = -\frac{b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a} = \overline{r_1}.$$

$$\Re(r_1) = -\frac{b}{2a} = \alpha$$

$$\Im(r_1) = \frac{\sqrt{4ac - b^2}}{2a} = \beta$$

Then the solution of the equation is

$$y(t) = e^{r_1 t} = e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

Then the fundamental solution set is

$$y_1(t) = \Re[y(t)] = e^{\alpha t}\cos\beta t, \qquad y_2(t) = \Im[y(t)] = e^{\alpha t}\sin\beta t$$

Let us find $W[y_1, y_2](t)$ $\neq 0$ (never zero)

If the auxiliary equation

$$ar^2 + br + c = 0$$

has two complex-conjugate roots

$$r_{1,2} = -\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a},$$

then the general solution of the equation

$$ay'' + by' + cy = 0$$

is

$$y(t) = e^{\alpha t}\left(C_1 \cos\beta t + C_2 \sin\beta t\right),$$

where $\alpha = -\dfrac{b}{2a}$, $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$, C_1 and C_2 are arbitrary constants.

Example 1. Find the general solution of the equation

$$y'' - 2y' + 2y = 0$$

auxiliary ~~equation~~ $r^2 - 2r + 2 = 0$

roots: $r_1 = \frac{2 + \sqrt{4 - 4(2)}}{2} = \frac{2 + \sqrt{-4}}{2} = \frac{2 + \sqrt{(-1) \cdot 4}}{2} = \frac{2 + 2i}{2} = 1 + i$

$r_2 = 1 - i$

$Re(r_1) = 1$

$Im(r_1) = 1$

general solution: $y(t) = c_1 e^{t} (c_1 \cos t + c_2 \sin t)$

Example 2. Solve the initial value problem

$$y'' + 4y' + 5y = 0, \qquad y(0) = 1, y'(0) = 0.$$

auxiliary equation: $r^2 + 4r + 5 = 0$

roots: $r_1 = \frac{-4+\sqrt{16-4(5)}}{2} = \frac{-4+\sqrt{16-20}}{2} = \frac{-4+\sqrt{-4}}{2} = \frac{-4+2i}{2} = -2+i$

$Re(r_1) = -2$

$Im(r_1) = 1$

general solution: $y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t)$

$y(0) = c_1 = 1$

$y'(t) = -2e^{-2t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(-c_1 \sin t + c_2 \cos t)$

$y'(0) = -2c_1 + c_2 = 0$

$c_2 = 2c_1 = 2$

$$\boxed{y(t) = e^{-2t}(\cos t + 2\sin t)}$$