Section 3.4 **Repeated roots; reduction of order.**

$$ay'' + by' + cy = 0$$

where a, b, c are constants and $b^2 - 4ac = 0$.

Then the auxiliary equation

$$ar^2 + br + c = 0$$

has one repeated root

$$r = -\frac{b}{2a}$$

and the corresponding solution of the equation is $y_1(t) = e^{-\frac{b}{2a}t}$.

We use **reduction of order** to find a second solution.

We'll look for the general solution of the form

$$y(t) = v(t)e^{-\frac{b}{2a}t},$$

where $v(t)$ is an unknown function. Let's plug $y(t)$ back into the equation:

$$y'(t) = v'e^{-\frac{b}{2a}t} - \frac{b}{2a}ve^{-\frac{b}{2a}t} = e^{-\frac{b}{2a}t}\left[v' - \frac{b}{2a}v\right]$$

$$y'' = v''e^{-\frac{b}{2a}t} - v'\frac{b}{2a}e^{-\frac{b}{2a}t} - \frac{b}{2a}v'e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}ve^{-\frac{b}{2a}t}$$

$$= e^{-\frac{b}{2a}t}\left[v'' - v'\frac{b}{a} + \frac{b^2}{4a^2}v\right]$$

$ay'' + by' + cy = 0, \quad b^2 - 4ac = 0.$

$$a\underbrace{e^{-\frac{b}{2a}t}\left[v'' - v'\frac{b}{a} + \frac{b^2}{4a^2}v\right]}_{y''} + b\underbrace{e^{-\frac{b}{2a}t}\left[v' - \frac{b}{2a}v\right]}_{y'} + c\underbrace{ve^{-\frac{b}{2a}t}}_{y} = 0$$

$$a\left[v'' - v'\frac{b}{a} + \frac{b^2}{4a^2}v\right] + b\left[v' - \frac{b}{2a}v\right] + cv = 0$$

$$av'' - \cancel{v'b} + \frac{b^2}{4a}v + \cancel{bv'} - \frac{b^2}{2a}v + cv = 0$$

$$av'' + v\frac{b^2 - 2b^2 + 4ac}{4a} = 0 \quad \text{or} \quad av'' + v\frac{\overbrace{4ac - b^2}^{0}}{\cancel{4a}} = 0$$

$$av'' = 0$$
$$av' = c_1$$
$$av = c_1t + c_2 \quad \text{or} \quad v(t) = \overbrace{\frac{c_1}{a}}^{c_3}t + \overbrace{\frac{c_2}{a}}^{c_4}$$

$$v(t) = c_3t + c_4$$

Then $\quad y(t) = (c_3t + c_4)e^{-\frac{b}{2a}t}$

$$= c_3\underbrace{te^{-\frac{b}{2a}t}}_{y_2(t)} + c_4\underbrace{e^{-\frac{b}{2a}t}}_{y_1(t)}$$

$$y_2(t) = te^{-\frac{b}{2a}t}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W\left[e^{-\frac{b}{2a}t}, te^{-\frac{b}{2a}t}\right] = \begin{vmatrix} e^{-\frac{b}{2a}t} & te^{-\frac{b}{2a}t} \\ -\frac{b}{2a}e^{-\frac{b}{2a}t} & e^{-\frac{b}{2a}t} - \frac{b}{2a}te^{-\frac{b}{2a}t} \end{vmatrix}$$

$$= e^{-\frac{b}{2a}t}\left[e^{-\frac{b}{2a}t} - \cancel{\frac{b}{2a}te^{-\frac{b}{2a}t}}\right] + \cancel{\frac{b}{2a}te^{-\frac{b}{2a}t}}e^{-\frac{b}{2a}t} = e^{-\frac{b}{a}t} \neq 0$$

Thus, $\left\{e^{-\frac{b}{2a}t}, te^{-\frac{b}{2a}t}\right\}$ is the fundamental solution set.

If the auxiliary equation $ar^2 + br + c = 0$ has one repeated root

$$r = -\frac{b}{2a},$$

then the general solution of the equation $ay'' + by' + cy = 0$ is

$$y(t) = e^{-\frac{b}{2a}t}\left(C_1 + C_2 t\right),$$

where C_1 and C_2 are arbitrary constants.

Example 1. Find the general solution of the equation

$$9y'' + 6y' + y = 0$$

auxiliary equation is $9r^2+6r+1=0$

$(3r+1)^2=0$

$r=-\frac{1}{3}$ repeated root

general solution $\boxed{y(t)=e^{-\frac{t}{3}}(C_1+C_2t)}$

Example 2. Solve the initial value problem

$$y'' - 4y' + 4y = 0, \qquad y(0) = 0, y'(0) = 2.$$

auxiliary equation $r^2-4r+4=0$

$(r-2)^2=0$

$r=2$ repeated root

general solution $y(t)=e^{2t}(C_1+C_2t)$ | $y(0)=C_1=0 \Rightarrow C_1=0,\ C_2=2$

$y'(t)=2e^{2t}(C_1+C_2t)+C_2e^{2t}$ | $y'(0)=2C_1+C_2=2$

$\boxed{y(t)=2te^{2t}}$