Section 3.5. **Nonhomogeneous equations; Method of undertermined coefficients**

Given a nonhomogeneous linear equation

$$y'' + p(t)y' + q(t)y = g(t)$$

Procedure for solving nonhomogeneous equations.

To solve $y'' + p(t)y' + q(t)y = g(t)$:

Step 1. Determine the general solution $c_1y_1(t) + c_2y_2(t)$ of the corresponding homogeneous equation.

Step 2. Find the particular solution $y_p(t)$ of the given nonhomogeneous equation.

Step 3. Form the sum of the particular solution and a general solution to the homogeneous equation

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t),$$

to obtain the general solution to the given equation.

Method of undertermined coefficients.

$$ay'' + by' + cy = g(t)$$

where $g(t)$ is of a special form. The corresponding auxiliary equation is

$$ar^2 + br + c = 0$$

1. $g(t) = e^{\alpha t}(p_0t^m + p_1t^{m-1} + p_2t^{m-2} + \ldots + p_{m-1}t + p_m)$. Then
   - $y_p(t) = e^{\alpha t}(At^m + Bt^{m-1} + Ct^{m-2} + \ldots + Dt + F)$, if $r = \alpha$ is **not** a root to the auxiliary equation.
   - $y_p(t) = e^{\alpha t}t(At^m + Bt^{m-1} + Ct^{m-2} + \ldots + Dt + F)$, if $r = \alpha$ is **one of two** roots of the auxiliary equation
   - $y_p(t) = e^{\alpha t}t^2(At^m + Bt^{m-1} + Ct^{m-2} + \ldots + Dt + F)$, if $r = \alpha$ is a **repeated** root to the auxiliary equation.

Example 1. Find the general solution to the following equations

(a) $y'' - 2y' = 2e^{-2t}$

(b) $y'' - 4y' + 4y = 16t^2 e^{2t}$

2. $g(t) = (p_0 t^{m_1} + p_1 t^{m_1-1} + p_2 t^{m_1-2} + \ldots + p_{m_1-1} t + p_{m_1}) \cos(\beta t) + (q_0 t^{m_2} + q_1 t^{m_2-1} + q_2 t^{m_2-2} + \ldots + q_{m_2-1} t + q_{m_2}) \sin(\beta t)$. Then

   - $y_p(t) = (A_0 t^m + A_1 t^{m-1} + A_2 t^{m-2} + \ldots + A_{m-1} t + A_m) \cos(\beta t) + (B_0 t^m + B_1 t^{m-1} + B_2 t^{m-2} + \ldots + B_{m-1} t + B_m) \sin(\beta t)$,
     if $i\beta$ is **not** a root to the auxiliary equation. Here $m = \max\{m_1, m_2\}$.
   - $y_p(t) = t(A_0 t^m + A_1 t^{m-1} + A_2 t^{m-2} + \ldots + A_{m-1} t + A_m) \cos(\beta t) + t(B_0 t^m + B_1 t^{m-1} + B_2 t^{m-2} + \ldots + B_{m-1} t + B_m) \sin(\beta t)$,
     if $i\beta$ is **one of two** roots of the auxiliary equation.

Example 2. Find the solution to the initial value problem

$$y'' - 3y' + 2y = 4t \sin t, \qquad y(0) = 3, \quad y'(0) = 2.$$

3. $g(t) = e^{\alpha t}[(p_0 t^{m_1} + p_1 t^{m_1-1} + p_2 t^{m_1-2} + \ldots + p_{m_1-1}t + p_{m_1})\cos(\beta t) + (q_0 t^{m_2} + q_1 t^{m_2-1} + q_2 t^{m_2-2} + \ldots + q_{m_2-1}t + q_{m_2})\sin(\beta t)]$.Then

- $y_p(t) = e^{\alpha t}[(A_0 t^m + A_1 t^{m-1} + A_2 t^{m-2} + \ldots + A_{m-1}t + A_m)\cos(\beta t) + (B_0 t^m + B_1 t^{m-1} + B_2 t^{m-2} + \ldots + B_{m-1}t + B_m)\sin(\beta t)]$,
  if $\alpha + i\beta$ is **not** a root to the auxiliary equation. Here $m = \max\{m_1, m_2\}$.
- $y_p(t) = e^{\alpha t}t[(A_0 t^m + A_1 t^{m-1} + A_2 t^{m-2} + \ldots + A_{m-1}t + A_m)\cos(\beta t) + (B_0 t^m + B_1 t^{m-1} + B_2 t^{m-2} + \ldots + B_{m-1}t + B_m)\sin(\beta t)]$,
  if $\alpha + i\beta$ is **one of two** roots of the auxiliary equation, then the particular solution is

Example 3. Find a general solution to the equation

$$y'' - 9y = e^{3t}\cos t.$$

Particular solutions to $ay'' + by' + cy = g(t)$

Type	$g(t)$	$y_p(t)$
I	$p_0 t^k + p_1 t^{k-1} + \ldots + p_k$	$t^s(At^k + Bt^{k-1} + ... + F)$
II	$de^{\alpha t}$	$t^s A e^{\alpha t}$
III	$e^{\alpha t}\left(p_0 t^k + ... + p_k\right)$	$t^s e^{\alpha t}\left(At^k + Bt^{k-1} + \ldots + F\right)$
IV	$d\cos(\beta t) + f\sin(\beta t)$	$t^s\left[A\cos(\beta t) + B\sin(\beta t)\right]$
V	$P_{m_1}(t)\cos(\beta t) + Q_{m_2}(t)\sin(\beta t)$	$t^s\left[(A_0 t^m + ... + A_m)\cos(\beta t) + (B_0 t^m + ... + B_m)\sin(\beta t)\right]$
VI	$e^{\alpha t}\left[d\cos(\beta t) + f\sin(\beta t)\right]$	$t^s e^{\alpha t}\left[A\cos(\beta t) + B\sin(\beta t)\right]$
VII	$e^{\alpha t}[P_{m_1}(t)\cos(\beta t) + Q_{m_2}(t)\sin(\beta t)]$	$t^s e^{\alpha t}[(A_0 t^m + ... + A_m)\cos(\beta t) + (B_0 t^m + ... + B_m)\sin(\beta t)]$

In this table

- $s = 0$, when $\alpha + i\beta$ is not a root to the auxiliary equation
- $s = 1$, when $\alpha + i\beta$ is one of two roots to the auxiliary equation
- $s = 0$, when $\beta = 0$ and α is a repeated root to the auxiliary equation;
- $m = \max\{m_1, m_2\}$
- $P_{m_1}(t) = p_0 t^{m_1} + p_1 t^{m_1 - 1} + p_2 t^{m_1 - 2} + \ldots + p_{m_1 - 1} t + p_{m_1}$
- $Q_{m_2}(t) = q_0 t^{m_2} + q_1 t^{m_2 - 1} + q_2 t^{m_2 - 2} + \ldots + q_{m_2 - 1} t + q_{m_2}$

Example 4. Using Table, find the form for a particular solution $y_p(t)$ to

$$y'' + y' - 2y = g(t),$$

where $g(t)$ equals
(a) $g(t) = (2t^2 + 3)e^{-2t}$

(b) $g(t) = t\sin 2t$

(c) $g(t) = e^t + \cos 3t$