Section 3.5. **Nonhomogeneous equations; Method of undertermined coefficients**

Given a nonhomogeneous linear equation

$$y'' + p(t)y' + q(t)y = g(t)$$

Procedure for solving nonhomogeneous equations.

To solve $y'' + p(t)y' + q(t)y = g(t)$:

Step 1. Determine the general solution $c_1y_1(t) + c_2y_2(t)$ of the corresponding homogeneous equation.

Step 2. Find the particular solution $y_p(t)$ of the given nonhomogeneous equation.

Step 3. Form the sum of the particular solution and a general solution to the homogeneous equation

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t),$$

to obtain the general solution to the given equation.

Method of undertermined coefficients.

$$ay'' + by' + cy = g(t)$$

where $g(t)$ is of a special form. The corresponding auxiliary equation is

$$ar^2 + br + c = 0$$

1. $g(t) = e^{\alpha t}(p_0 t^m + p_1 t^{m-1} + p_2 t^{m-2} + \ldots + p_{m-1} t + p_m)$. Then
   - $y_p(t) = e^{\alpha t}(At^m + Bt^{m-1} + Ct^{m-2} + \ldots + Dt + F)$, if $r = \alpha$ is **not** a root to the auxiliary equation.
   - $y_p(t) = e^{\alpha t} t(At^m + Bt^{m-1} + Ct^{m-2} + \ldots + Dt + F)$, if $r = \alpha$ is **one of two** roots of the auxiliary equation
   - $y_p(t) = e^{\alpha t} t^2(At^m + Bt^{m-1} + Ct^{m-2} + \ldots + Dt + F)$, if $r = \alpha$ is a **repeated** root to the auxiliary equation.

Example 1. Find the general solution to the following equations

(a) $y'' - 2y' = 2e^{-2t}$

corresponding homogeneous equation: $y'' - 2y' = 0$

auxiliary equation: $r^2 - 2r = 0$

$r(r-2) = 0 \Rightarrow r_1 = 0,\ r_2 = 2$

general solution of the homogeneous equation is: $y_h(t) = C_1 e^{0 \cdot t} + C_2 e^{2t}$

$\boxed{y_h(t) = C_1 + C_2 e^{2t}}$

particular solution of the nonhomogeneous equation

$y_p(t) = Ae^{-2t}$, A is an unknown constant.

Plug $y_p(t)$ back into the nonhomogeneous equation.

$y_p'(t) = -2Ae^{-2t}$, $y_p''(t) = 4Ae^{-2t}$

$\underbrace{4A\cancel{e^{-2t}}}_{y_p''} - 2\underbrace{(-2A\cancel{e^{-2t}})}_{y_p'} = 2\cancel{e^{-2t}}$

$8A = 2$ or $\boxed{A = \frac{1}{4}}$

$\boxed{y_p(t) = \frac{1}{4}e^{-2t}}$

General solution of the nonhomogeneous equation: $\boxed{y(t) = C_1 + C_2 e^{2t} + \frac{1}{4}e^{-2t}}$

(b) $y'' - 4y' + 4y = 16t^2 e^{2t}$

corresponding homogeneous equation: $y'' - 4y' + 4y = 0$

auxiliary equation: $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$ $\quad r = 2$ repeated root

general solution of the homogeneous equation: $\boxed{y_h(t) = (C_1 + C_2 t)e^{2t}}$

Particular solution of the nonhomogeneous equation:

$$y_p(t) = e^{2t}\underbrace{(At^2 + Bt + C)}_{\text{polynomial of degree 2 with unknown coefficients}} t^2$$

2 is the repeated root of the auxiliary equation.

$$y_p(t) = e^{2t}(At^4 + Bt^3 + Ct^2)$$

$$y_p'(t) = 2e^{2t}(At^4 + Bt^3 + Ct^2) + e^{2t}(4At^3 + 3Bt^2 + 2Ct)$$

$$y_p''(t) = 4e^{2t}(At^4 + Bt^3 + Ct^2) + 2e^{2t}(4At^3 + 3Bt^2 + 2Ct) + 2e^{2t}(4At^3 + 3Bt^2 + 2Ct) + e^{2t}(12At^2 + 6Bt + 2C)$$

$$y_p''(t) = 4e^{2t}(At^4 + Bt^3 + Ct^2) + 4e^{2t}(4At^3 + 3Bt^2 + 2Ct) + e^{2t}(12At^2 + 6Bt + 2C)$$

$$4e^{2t}(At^4 + Bt^3 + Ct^2) + 4e^{2t}(4At^3 + 3Bt^2 + 2Ct) + e^{2t}(12At^2 + 6Bt + 2C)$$
$$-4\left[2e^{2t}(At^4 + Bt^3 + Ct^2) + e^{2t}(4At^3 + 3Bt^2 + 2Ct)\right]$$
$$+4e^{2t}(At^4 + Bt^3 + Ct^2) = 16t^2 e^{2t}$$

$$\cancel{e^{2t}}(12At^2 + 6Bt + 2C) = 16t^2 \cancel{e^{2t}}$$

Equate coefficients to the corresponding powers of t:

t^2: $12A = 16$ $\quad A = \frac{16}{12} = \frac{4}{3}$

t: $6B = 0$ $\quad B = 0$

1: $2C = 0$ $\quad C = 0$

$$\boxed{y_p(t) = \frac{4}{3}e^{2t} \cdot t^4}$$

General solution of the nonhomogeneous equation: $\boxed{y(t) = (C_1 + C_2 t)e^{2t} + \frac{4}{3}t^4 e^{2t}}$

2. $g(t) = (p_0 t^{m_1} + p_1 t^{m_1-1} + p_2 t^{m_1-2} + \ldots + p_{m_1-1}t + p_{m_1})\cos(\beta t) + (q_0 t^{m_2} + q_1 t^{m_2-1} + q_2 t^{m_2-2} + \ldots + q_{m_2-1}t + q_{m_2})\sin(\beta t)$. Then

- $y_p(t) = (A_0 t^m + A_1 t^{m-1} + A_2 t^{m-2} + \ldots + A_{m-1}t + A_m)\cos(\beta t) + (B_0 t^m + B_1 t^{m-1} + B_2 t^{m-2} + \ldots + B_{m-1}t + B_m)\sin(\beta t)$,
  if $i\beta$ is **not** a root to the auxiliary equation. Here $m = \max\{m_1, m_2\}$.
- $y_p(t) = t(A_0 t^m + A_1 t^{m-1} + A_2 t^{m-2} + \ldots + A_{m-1}t + A_m)\cos(\beta t) + t(B_0 t^m + B_1 t^{m-1} + B_2 t^{m-2} + \ldots + B_{m-1}t + B_m)\sin(\beta t)$,
  if $i\beta$ is **one of two** roots of the auxiliary equation.

Example 2. Find the solution to the initial value problem

$$y'' - 3y' + 2y = 4t\sin t, \qquad y(0) = 3, \quad y'(0) = 2.$$

homogeneous equation: $y'' - 3y' + 2y = 0$

auxiliary equation: $r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0 \Rightarrow r_1 = 2,\ r_2 = 1$

general solution of the auxiliary equation $\boxed{y_h(t) = C_1 e^t + C_2 e^{2t}}$

Particular solution of the nonhomogeneous equation

$y_p(t) = (At+B)\sin t + (Ct+D)\cos t$

$y_p'(t) = A\sin t + (At+B)\cos t + C\cos t - (Ct+D)\sin t$

$y_p''(t) = A\cos t + A\cos t - (At+B)\sin t - C\sin t - C\sin t - (Ct+D)\cos t$

$= 2A\cos t - (At+B)\sin t - 2C\sin t - (Ct+D)\cos t$

$2A\cos t - (At+B)\sin t - 2C\sin t - (Ct+D)\cos t - 3\left[A\sin t + (At+B)\cos t + C\cos t - (Ct+D)\sin t\right]$

$+ 2(At+B)\sin t + 2(Ct+D)\cos t = 4t\sin t$

$$\left.\begin{aligned} t\cos t &: -C - 3A + 2C = 0 \\ \cos t &: 2A - D - 3B - 3C + 2D = 0 \\ t\sin t &: -A + 3C + 2A = 4 \\ \sin t &: -B - 2C - 3A + 3D + 2B = 0 \end{aligned}\right\}$$

$\begin{cases} C - 3A = 0 \Rightarrow C = 3A \\ A + 3C = 4 \Rightarrow 10A = 4 \Rightarrow \end{cases}$ $\boxed{A = \frac{4}{10} = \frac{2}{5}}$ $\boxed{C = \frac{6}{5}}$

$\begin{cases} D - 3B = 3C - 2A = \frac{18}{5} - \frac{4}{5} = \frac{14}{5} \\ B + 3D = 2C + 3A = \frac{12}{5} + \frac{6}{5} = \frac{18}{5} \end{cases} \Rightarrow \begin{cases} D - 3B = \frac{14}{5} \\ B + 3D = \frac{18}{5} \end{cases} \Rightarrow D = 3B + \frac{14}{5}$

$B + 9B + \frac{42}{5} = \frac{18}{5}$

$10B = -\frac{24}{5}$, $\boxed{B = -\frac{24}{50} = -\frac{12}{25}}$

$D = \frac{-36}{25} + \frac{14}{5} = \boxed{\frac{34}{25} = D}$

$\boxed{y_p(t) = \left(\frac{2}{5}t - \frac{12}{25}\right)\sin t + \left(\frac{6}{5}t + \frac{34}{25}\right)\cos t}$

General solution of the nonhomogeneous equation:

$\boxed{y(t) = C_1 e^t + C_2 e^{2t} + \left(\frac{2}{5}t - \frac{12}{25}\right)\sin t + \left(\frac{6}{5}t + \frac{34}{25}\right)\cos t}$

Example.

Find the form of a particular solution for the equation

$$y'' - 2y' - 3y = g(t), \text{ where}$$

corresponding homogeneous equation: $y'' - 2y' - 3y = 0$

auxiliary equation: $r^2 - 2r - 3 = 0$

$r_1 = 3,\ r_2 = -1$

general solution of the homogeneous equation

$$y_h(t) = C_1 e^{-t} + C_2 e^{3t}$$

$t^2 \longrightarrow At^2 + Bt + C$

$t \longrightarrow At + B$

$4 \longrightarrow A$

$t^3 \longrightarrow At^3 + Bt^2 + Ct + D$

1. $g(t) = t^2 e^t$

$y_p(t) = e^t(At^2 + Bt + C)$

2. $g(t) = t^3 e^{-t}$

$y_p(t) = e^{-t}(At^3 + Bt^2 + Ct + D)t$

3. $g(t) = \cos t + t^2 \sin t$

$y_p(t) = \cos t\,(At^2 + Bt + C) + \sin t\,(Dt^2 + Et + F)$

$$y'' + 2y' + y = 4e^{-t}$$

homogeneous equation $y'' + 2y' + y = 0$

auxiliary equation $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0 \Rightarrow r = -1$ repeated root

$y_h(t) = (C_1 + C_2 t)e^{-t}$

$y_p(t) = Ae^{-t} t^2$

$g(t) = Ae^{rt}$

homogeneous equation	partiular solution
$y_h(t) = C_1 e^{rt} + C_2 e^{st}$	$y_p(t) = t(\dots)e^{rt}$
$y_h(t) = C_1 e^{rt} + C_2 t e^{rt}$	$y_p(t) = t^2(\dots)e^{rt}$

$y'' + 9y = g(t)$

$\boxed{\begin{array}{l} y'' \to r^2 \\ y' \to r \\ y \to 1 \end{array}}$

homogeneous equation: $y'' + 9y = 0$

auxiliary equation $r^2 + 9 = 0$, $r = \pm 3i$, $Re(r) = 0$, $Im(r) = \pm 3$

$y_h(t) = C_1 \cos 3t + C_2 \sin 3t$

1. $g(t) = te^{3t}$

$y_p(t) = (At + B)e^{3t}$

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2. $g(t) = 5\sin 2t$

$y_p(t) = A\sin 2t + B\cos 2t$

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3. $g(t) = t\cos 3t$

$y_p(t) = \left[(At + B)\cos 3t + (Ct + D)\sin 3t\right]t$

3. $g(t) = e^{\alpha t}[(p_0 t^{m_1} + p_1 t^{m_1-1} + p_2 t^{m_1-2} + \ldots + p_{m_1-1} t + p_{m_1})\cos(\beta t) + (q_0 t^{m_2} + q_1 t^{m_2-1} + q_2 t^{m_2-2} + \ldots + q_{m_2-1} t + q_{m_2})\sin(\beta t)]$. Then

- $y_p(t) = e^{\alpha t}[(A_0 t^m + A_1 t^{m-1} + A_2 t^{m-2} + \ldots + A_{m-1} t + A_m)\cos(\beta t) + (B_0 t^m + B_1 t^{m-1} + B_2 t^{m-2} + \ldots + B_{m-1} t + B_m)\sin(\beta t)]$,

  if $\alpha + i\beta$ is **not** a root to the auxiliary equation. Here $m = \max\{m_1, m_2\}$.
- $y_p(t) = e^{\alpha t} t[(A_0 t^m + A_1 t^{m-1} + A_2 t^{m-2} + \ldots + A_{m-1} t + A_m)\cos(\beta t) + (B_0 t^m + B_1 t^{m-1} + B_2 t^{m-2} + \ldots + B_{m-1} t + B_m)\sin(\beta t)]$,

  if $\alpha + i\beta$ is **one of two** roots of the auxiliary equation, then the particular solution is

Example 3. Find ~~a~~ the general solution of the equation

$$y'' - 9y = e^{3t}\cos t.$$

homogeneous equation $y'' - 9y = 0$

auxiliary equation $r^2 - 9 = 0$ or $r = \pm 3$

$$y_h(t) = C_1 e^{-3t} + C_2 e^{3t}$$

Particular solution:

$$y_p(t) = Ae^{3t}\cos t + Be^{3t}\sin t = e^{3t}(A\cos t + B\sin t)$$

$$y_p'(t) = 3e^{3t}(A\cos t + B\sin t) + e^{3t}(-A\sin t + B\cos t)$$

$$y_p''(t) = 9e^{3t}(A\cos t + B\sin t) + 3e^{3t}(-A\sin t + B\cos t) + 3e^{3t}(-A\sin t + B\cos t) + e^{3t}(-A\cos t - B\sin t)$$

$$y_p''(t) = 9e^{3t}(A\cos t + B\sin t) + 6e^{3t}(-A\sin t + B\cos t) + e^{3t}(-A\cos t - B\sin t)$$

$$\cancel{9e^{3t}(A\cos t + B\sin t)} + 6\cancel{e^{3t}}(-A\sin t + B\cos t) + \cancel{e^{3t}}(-A\cos t - B\sin t) - 9\cancel{e^{3t}(A\cos t + B\sin t)} = \cancel{e^{3t}}\cos t$$

$\cos t$: $6B - A = 1$

$\sin t$: $-6A - B = 0 \Rightarrow B = -6A$

$-37A = 1 \Rightarrow \boxed{A = -\frac{1}{37}} \quad \boxed{B = \frac{6}{37}}$

$$y_p = \frac{e^{3t}}{37}(-\cos t + 6\sin t)$$

General solution $\boxed{y(t) = C_1 e^{-3t} + C_2 e^{3t} + \frac{e^{3t}}{37}(-\cos t + 6\sin t)}$

Particular solutions to $ay'' + by' + cy = g(t)$

Type	$g(t)$	$y_p(t)$
I	$p_0t^k + p_1t^{k-1} + \ldots + p_k$	$t^s(At^k + Bt^{k-1} + ... + F)$
II	$de^{\alpha t}$	$t^s Ae^{\alpha t}$
III	$e^{\alpha t}\left(p_0t^k + ... + p_k\right)$	$t^s e^{\alpha t}\left(At^k + Bt^{k-1} + \ldots + F\right)$
IV	$d\cos(\beta t) + f\sin(\beta t)$	$t^s\left[A\cos(\beta t) + B\sin(\beta t)\right]$
V	$P_{m_1}(t)\cos(\beta t) + Q_{m_2}(t)\sin(\beta t)$	$t^s\left[(A_0t^m + ... + A_m)\cos(\beta t) + (B_0t^m + ... + B_m)\sin(\beta t)\right]$
VI	$e^{\alpha t}\left[d\cos(\beta t) + f\sin(\beta t)\right]$	$t^s e^{\alpha t}\left[A\cos(\beta t) + B\sin(\beta t)\right]$
VII	$e^{\alpha t}[P_{m_1}(t)\cos(\beta t) + Q_{m_2}(t)\sin(\beta t)]$	$t^s e^{\alpha t}[(A_0t^m + ... + A_m)\cos(\beta t) + (B_0t^m + ... + B_m)\sin(\beta t)]$

In this table

- $s = 0$, when $\alpha + i\beta$ is not a root to the auxiliary equation
- $s = 1$, when $\alpha + i\beta$ is one of two roots to the auxiliary equation
- $s = 0$, when $\beta = 0$ and α is a repeated root to the auxiliary equation;
- $m = \max\{m_1, m_2\}$
- $P_{m_1}(t) = p_0t^{m_1} + p_1t^{m_1-1} + p_2t^{m_1-2} + \ldots + p_{m_1-1}t + p_{m_1}$
- $Q_{m_2}(t) = q_0t^{m_2} + q_1t^{m_2-1} + q_2t^{m_2-2} + \ldots + q_{m_2-1}t + q_{m_2}$

Example 4. Using Table, find the form for a particular solution $y_p(t)$ to

$$y'' + y' - 2y = g(t),$$

where $g(t)$ equals

(a) $g(t) = (2t^2 + 3)e^{-2t} + t$

$y_p(t) = e^{-2t}(At^2 + Bt + C)t + (Dt + E)$

homogeneous equation: $y'' + y' - 2y = 0$

auxiliary equation: $r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$r_1 = -2, \ r_2 = 1$

$y_h(t) = C_1 e^{-2t} + C_2 e^{t}$

(b) $g(t) = t \sin \underline{\underline{2}}t + \cos \underline{\underline{3}}t$

$y_p(t) = (At + B)\sin 2t + (Ct + D)\cos 2t + E\cos 3t + F\sin 3t$

(c) $g(t) = e^t + \cos 3t$

$y_p(t) = e^t \cdot A \cdot t + B\cos 3t + C\sin 3t$