## Section 3.6 **Variation of Parameters**

Consider the nonhomogeneous linear second order differential equation

$$y'' + p(x)y' + q(x)y = g(x). \tag{1}$$

Let $\{y_1(x), y_2(x)\}$ be a fundamental solution set to the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

The general solution to this homogeneous equation is $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$, where $c_1$ and $c_2$ are constants. To find a particular solution to (1) we assume that $c_1 = c_1(x)$ and $c_2 = c_2(x)$ are functions of $x$ and we seek a particular solution $y_p(x)$ in the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

Let's substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into (1):

To summarize, we can find $c_1(x)$ and $c_2(x)$ solving the system

$$\begin{cases} c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = g(x) \end{cases}$$

for $c_1'(x)$ and $c_2'(x)$. Cramer's rule gives

$$c_1'(x) = \frac{-g(x)y_2(x)}{W[y_1, y_2](x)}, \qquad c_2'(x) = \frac{g(x)y_1(x)}{W[y_1, y_2](x)}.$$

Then

$$\boxed{c_1(x) = \int \frac{-g(x)y_2(x)}{W[y_1, y_2](x)}dx} \qquad \boxed{c_2(x) = \int \frac{g(x)y_1(x)}{W[y_1, y_2](x)}dx}$$

**Example 1.** Find the general solution to the equation

$$y'' + y = \frac{1}{\sin x}.$$