Section 3.6 **Variation of Parameters**

Consider the nonhomogeneous linear second order differential equation

$$y'' + p(x)y' + q(x)y = g(x). \tag{1}$$

Let $\{y_1(x), y_2(x)\}$ be a fundamental solution set to the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

The general solution to this homogeneous equation is $\boxed{y_h(x) = c_1y_1(x) + c_2y_2(x)}$, where c_1 and c_2 are constants. To find a particular solution to (1) we assume that $c_1 = c_1(x)$ and $c_2 = c_2(x)$ are functions of x and we seek a particular solution $y_p(x)$ in the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

Let's substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into (1):

$$y_p' = \underbrace{c_1'(x)\,y_1(x) + c_2'(x)\,y_2(x)}_{0} + c_1(x)\,y_1'(x) + c_2(x)\,y_2'(x)$$

$$y_p' = c_1(x)\,y_1'(x) + c_2(x)\,y_2'(x)$$

$$y_p'' = c_1'(x)\,y_1'(x) + c_1(x)\,y_1''(x) + c_2'(x)\,y_2'(x) + c_2(x)\,y_2''(x)$$

$$y'' + p(x)\,y' + q(x) = g(x)$$

$$\underbrace{c_1'(x)\,y_1'(x) + c_1(x)\,y_1''(x) + c_2'(x)\,y_2'(x) + c_2(x)\,y_2''(x)}_{y_p''} + p(x)\underbrace{\left[c_1(x)\,y_1'(x) + c_2(x)\,y_2'(x)\right]}_{y_p'} + q(x)\underbrace{\left[c_1(x)\,y_1(x) + c_2(x)\,y_2(x)\right]}_{y_p} = g(x)$$

$$c_1'(x)\,y_1'(x) + c_2'(x)\,y_2'(x) + c_1(x)\underbrace{\left[y_1'' + p(x)\,y_1' + q(x)\right]}_{0} + c_2(x)\underbrace{\left[y_2'' + p(x)\,y_2' + q(x)\right]}_{0} = g(x)$$

$$c_1'(x)\,y_1'(x) + c_2'(x)\,y_2'(x) = g(x)$$

To summarize, we can find $c_1(x)$ and $c_2(x)$ solving the system

$$\boxed{\begin{cases} c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = g(x) \end{cases}}$$

for $c_1'(x)$ and $c_2'(x)$. Cramer's rule gives

$$c_1'(x) = \frac{-g(x)y_2(x)}{W[y_1, y_2](x)}, \qquad c_2'(x) = \frac{g(x)y_1(x)}{W[y_1, y_2](x)}.$$

Then

$$\boxed{c_1(x) = \int \frac{-g(x)y_2(x)}{W[y_1, y_2](x)} dx} \quad \boxed{c_2(x) = \int \frac{g(x)y_1(x)}{W[y_1, y_2](x)} dx}$$

Example 1. Find the general solution of the equation

$$y'' + y = \frac{1}{\sin x}.$$

Homogeneous equation: $y''+y=0$

auxiliary equation: $r^2+1=0$, $r=\pm i$

$$y_h(x) = c_1 \underbrace{\cos x}_{y_1(x)} + c_2 \underbrace{\sin x}_{y_2(x)}$$

General solution of the nonhomogeneous equation

$$y(x) = c_1(x)\cos x + c_2(x)\sin x$$

where $c_1(x)$ and $c_2(x)$ are unknown functions.

$$W[y_1,y_2] = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$c_1(x) = -\int \frac{g(x)y_2(x)}{W[y_1,y_2]} dx = -\int \frac{\frac{1}{\sin x}\sin x}{1} dx = -\int 1\, dx = -x + C_3$$

$$c_2(x) = \int \frac{g(x)y_1(x)}{W[y_1,y_2]} dx = \int \frac{\frac{1}{\sin x}\cos x}{1} dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C_4$$

$$\boxed{y(x) = (-x+C_3)\cos x + (\ln|\sin x| + C_4)\sin x}$$