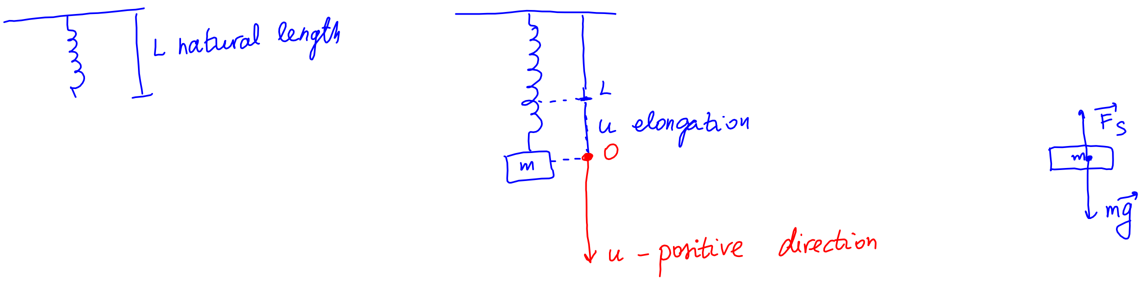
Section 3.6 **Mechanical and electrical vibrations**

A damped mass-spring oscillator consists of a mass m attached to a spring fixed at one end.



We study the motion of the mass when it is acted on an external force or is initially displaced. There are four separate forces we need to consider:

- The weight $w = mg$ of the mass (always directed downward).
- The spring force F_s is assumed to be proportional to the total elongation $L + u$ of the spring and always acts to restore the spring to its natural position.

$$F_s = -k(L + u)$$

- The damping of resistive force F_d always acts in the direction opposite to the direction of motion of mass. We assume, that the resistive force is proportional to the speed $\frac{du}{dt}$ of the mass.

$$F_d = -\gamma u'(t)$$

where $\gamma > 0$ is the damping constant.

Model for the motion of the mass is expressed by the initial value problem

$$\boxed{mu'' + \gamma u' + ku = F_{\text{external}},}\ \boxed{u(0) = u_0, \quad u'(0) = v_0,}$$

where m is a mass, γ is the damping coefficient, k is the stiffness.

Let's $F_{\text{external}} = 0$

1. **Undamped free case:** $\gamma = 0$

   The equation reduces to

   $$mu'' + ku = 0 | : m$$

   $$u'' + \omega^2 u = 0 \,,$$ auxiliary equation $r^2 + \omega^2 = 0$, $r = \pm i\omega$

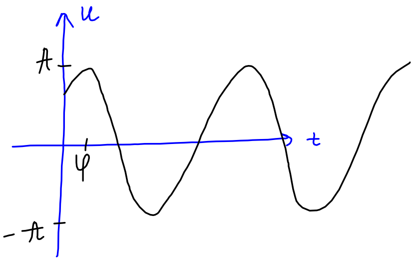
   where $\omega = \sqrt{\frac{k}{m}}$. The solution of this equation is

   $$u(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

   $$u(t) = A \sin(\omega t + \phi),$$

   where $A = \sqrt{C_1^2 + C_2^2}$, $\tan \phi = \frac{C_2}{C_1}$.

This motion is periodic with the amplitude A, phase φ, natural frequency $\omega = \sqrt{\frac{k}{m}}$, period $T = \frac{2\pi}{\omega}$.



Example 1. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in. then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t.

a) Set up an initial value problem for this system.

$mu'' + ku = 0$

$w = mg = 3 \Rightarrow \boxed{m = \frac{3}{g} = \frac{3}{32}}$

$F = kx$

$3 = \frac{3}{12}k$

$\boxed{k = 12}$

$3\text{in} = \frac{3}{12}\text{ft}$

$\frac{3}{32}u'' + 12u = 0$

$\boxed{u'' + 128u = 0, \quad u(0) = -\frac{1}{12}, \quad u'(0) = 2}$

b) Find the equation of motion of the mass.

auxiliary equation

$r^2 + 128 = 0$

$r^2 = -128$

$r = \pm\sqrt{128}\, i$

$r = \pm 8\sqrt{2}\, i$

$\sqrt{128} = \sqrt{4 \cdot 32} = \sqrt{4 \cdot 4 \cdot 4 \cdot 2} = 8\sqrt{2}$

general solution $y(t) = C_1 \cos(8\sqrt{2}t) + C_2 \sin(8\sqrt{2}t)$, $y'(t) = -C_1 8\sqrt{2} \sin(8\sqrt{2}t) + C_2 8\sqrt{2} \cos(8\sqrt{2}t)$

$y(0) = \boxed{C_1 = -\frac{1}{12}}$, $y'(0) = C_2\, 8\sqrt{2} = 2 \Rightarrow \boxed{C_2 = \frac{2}{8\sqrt{2}} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8}}$

$$\boxed{y(t) = -\frac{1}{12}\cos(8\sqrt{2}t) + \frac{\sqrt{2}}{8}\sin(8\sqrt{2}t)}$$

$$\boxed{y(t) = \frac{\sqrt{22}}{24}\sin(8\sqrt{2}t + \varphi)}$$

c) Find the amplitude, period and frequency of the motion.

frequency $\omega = 8\sqrt{2}$

period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8\sqrt{2}} = \frac{\pi}{4\sqrt{2}} = \frac{\sqrt{2}\pi}{8}$

amplitude $A = \sqrt{C_1^2 + C_2^2} = \sqrt{\frac{1}{144} + \frac{2}{64}} = \sqrt{\frac{4+18}{16 \cdot 9 \cdot 4}} = \frac{\sqrt{22}}{4 \cdot 3 \cdot 2} = \frac{\sqrt{22}}{24}$

$144 = 4 \cdot 3 \cdot 4 \cdot 3 = 16 \cdot 9$

$64 = 8 \cdot 8 = 16 \cdot 4$

phase $\varphi = \arctan\left(\frac{\frac{\sqrt{2}}{8}}{-\frac{1}{12}}\right) = \arctan\left(-\frac{3\sqrt{2}}{2}\right)$ II quadrant.

2. **Underdamped or oscillatory motion** ($\gamma^2 < 4mk$)

The solution to the equation

$$mu'' + \gamma u' + ku = 0 \quad , \quad \text{auxiliary equation} \quad mr^2 + \gamma r + k = 0.$$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \underbrace{-\frac{\gamma}{2m}}_{\alpha} \pm i \underbrace{\frac{\sqrt{4mk-\gamma^2}}{2m}}_{\beta}$$

is

$$u(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) = Ae^{\alpha t} \sin(\beta t + \phi), \longrightarrow 0 \text{ as } t \to \infty$$

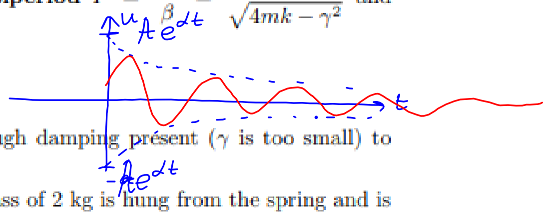
where

$$\alpha = -\frac{\gamma}{2m},\ \beta = \frac{1}{2m}\sqrt{4mk - \gamma^2},\ A = \sqrt{C_1^2 + C_2^2},\ \tan \phi = \frac{C_2}{C_1}.$$

The solution $u(t)$ varies between $-Ae^{\alpha t}$ and $Ae^{\alpha t}$ with **quasiperiod** $P = \frac{2\pi}{\beta} = \frac{4\pi m}{\sqrt{4mk - \gamma^2}}$ and

$\frac{\sqrt{4mk-\gamma^2}}{2m} =$ **quasifrequency** $\cancel{\beta}$. $u(t) \to 0$ as $t \to \infty$.

An exponential factor $Ae^{\alpha t}$ is called a **damping factor**.



The system is called **underdamped** because there is not enough damping present (γ is too small) to prevent the system from oscillating.

Example 2. A spring is stretch 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial velocity of 10 cm/s, determine its position u at any time. Find the quasifrequency of the motion.

$m = 2kg$

$F_d = \gamma v$

$3 = 5\gamma \quad , \gamma = \frac{3}{5}$

$F_s = kx$

$3 = k \cdot 0.1$

$k = 30$

$mu'' + \gamma u' + ku = 0$

$2u'' + \frac{3}{5}u' + 30u = 0$

$\boxed{u'' + 0.3u' + 15u = 0, \quad u(0) = 0.05, \quad u'(0) = 0.1}$

auxiliary equation $r^2 + 0.3k + 15 = 0$

$$r_1 = \frac{-0.3 + \sqrt{0.09 - 60}}{2} = -0.15 + \frac{\sqrt{-59.91}}{2}$$

$$r_1 = -0.15 + i\frac{\sqrt{59.91}}{2}, \quad r_2 = \overline{r_1}$$

$$u(t) = e^{-0.15t}\left[C_1 \cos\left(\frac{\sqrt{59.91}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{59.91}}{2}t\right)\right]$$

$u(0) = \boxed{C_1 = 0.05}$

$$u'(t) = -0.15e^{-0.15t}\left[C_1 \cos\left(\frac{\sqrt{59.91}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{59.91}}{2}t\right)\right] + e^{-0.15t}\left[-C_1\frac{\sqrt{59.91}}{2}\sin\left(\frac{\sqrt{59.91}}{2}t\right) + C_2\frac{\sqrt{59.91}}{2}\cos\left(\frac{\sqrt{59.91}}{2}t\right)\right]$$

$$u'(0) = -0.15C_1 + C_2\frac{\sqrt{59.91}}{2} = 0.1$$

$$C_2 = \frac{2}{\sqrt{59.91}}(0.1 + 0.15C_1) = \boxed{\frac{0.215}{\sqrt{59.91}} = C_2}$$

$$\boxed{\begin{gathered} u(t) = e^{-0.15t}\left[0.05\cos\left(\frac{\sqrt{59.91}}{2}t\right) + \frac{0.215}{\sqrt{59.91}}\sin\left(\frac{\sqrt{59.91}}{2}t\right)\right] \\ \text{quasifrequency} = \frac{\sqrt{59.91}}{2} \end{gathered}}$$

if the system was undamped ($\gamma = 0$), the the natural frequency would be $\omega = \sqrt{\frac{k}{m}} = \sqrt{15}$

3. **Overdamped motion** ($\gamma^2 > 4mk$)

The solution to the equation

$$mu'' + \gamma u' + ku = 0$$

, auxiliary equation

$mr^2 + \gamma r + k = 0$

$r_{1,2} = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

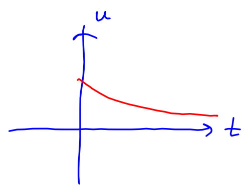
$r_1 < 0, \ r_2 < 0$

is

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

where $r_1 = -\dfrac{\gamma}{2m} + \dfrac{1}{2m}\sqrt{4mk - \gamma^2}$, $r_2 = -\dfrac{\gamma}{2m} - \dfrac{1}{2m}\sqrt{4mk - \gamma^2}$

$r_2 < 0$, and since $\gamma^2 > \gamma^2 - 4mk$, $r_1 < 0$. $u(t) \to 0$ as $t \to \infty$.

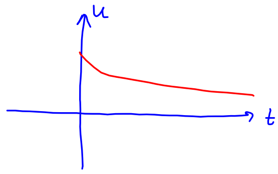


4. **Critically damped motion** ($\gamma^2 = 4mk$) The solution to the equation

$$mu'' + \gamma u' + ku = 0$$

is

$$u(t) = (c_1 + c_2 t)e^{-\frac{\gamma}{2m}t}.$$

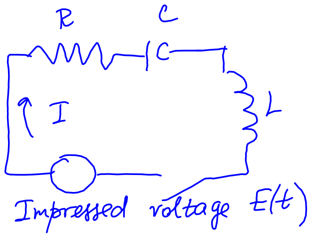


$u(t) \to 0$ as $t \to \infty$.

Electric circuits.

If Q is the charge at time t in an electrical closed circuit with inductance L, resistance R, and capacitance C, then by Kirchhoff's Second Law (from Physics) the impressed voltage $E(t)$ is equal to the sum of the voltage drops in the rest of the circuit

$$E(t) = IR + \frac{Q}{C} + LI'(t)$$



Bu substitution $I = Q'$ we get

$$\boxed{LQ'' + RQ' + \frac{1}{C}Q = E(t)}$$

Analogy between electrical and mechanical quantities:

Charge Q	Position u
Inductance L	mass m
Resistance R	Damping constant γ
Inverse capacitance $1/C$	Spring constant k
Impressed voltage $E(t)$ (electromotive force)	External force $F(t)$