## Section 3.8 **Forced vibrations**

Let's investigate the effect of a cosine forcing function on the system governed by the differential equation

$$mu'' + \gamma u' + ku = F_0 \cos \omega t,$$

where $F_0, \gamma$ are nonnegative constants and $\gamma^2 < 4mk$ (the system is underdamped).

The general solution to this equation is

$$u(t) = A\mathrm{e}^{-(\gamma/2m)t} \sin\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t + \phi\right) + \frac{F_0}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}} \sin(\omega t + \theta),$$

where $A, \phi$ are constants and $\tan\theta = \frac{k-m\gamma^2}{b\gamma}$.

$$y_h(t) = A\mathrm{e}^{-(\gamma/2m)t} \sin\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t + \phi\right)$$

is called the **transient part** of solution. $y_h \to 0$ as $t \to \infty$.

$$y_p(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}} \sin(\omega t + \theta)$$

is the offspring of the external forcing function $f(t) = F_0 \cos \omega t$. $y_p$ is sinusoidal with angular frequency $\omega$. $y_p$ is out of phase with $f(t)$ by the angle $\theta - \pi/2$, and its magnitude is different by the factor

$$\frac{1}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}$$

$y_p$ is called the **steady-state** solution.

The factor $\frac{1}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}$ is called the **frequency gain** or **gain factor**.

**Example 1.** A 2-kg mass is attached to a spring with stiffness $k = 45$ N/m. At time $t = 0$, an external force $f(t) = 12 \cos 3t$ is applied to the system. The damping constant for the system is 4 N-sec/m. Determine the steady-state solution for the system.

In general, the amplitude of the steady-state solution depends on $\omega$ and is given by

$$A(\omega) = F_0 M(\omega) = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

$M(\omega)$ is the frequency gain. The graph of $M(\omega)$ is called the **frequency response curve** or **resonance curve** for the system.

$M'(\omega) = 0$ when either $\omega = 0$ or

$$\omega = \omega_r = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}$$

When the system is critically damped or overdamped, $M'(\omega) = 0$ only when $\omega = 0$. When $\gamma^2 - 2mk < 0$, then $M'(\omega) = 0$ at

$$\omega_r = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}$$

The value $\omega_r/2\pi$ is called the **resonance frequency** for the system. When the system is stimulated by an external force as this frequency, it is said to be **at resonance**.

Consider the undamped system ($\gamma = 0$) with forcing term $F_0 \cos \omega t$. This system is governed by

$$m\frac{d^2y}{dt^2} + ky = F_0 \cos \omega t$$

$$y(t) = y_h(t) + y_p(t)$$

$$y_h(t) = A \sin(\omega_0 t + \phi), \qquad \omega_0 = \sqrt{k/m}$$

If $\omega \neq \omega_0$

$$y_p(t) = \frac{F_0}{k - m\omega^2} \sin(\omega t + \theta)$$

**Example 2.** The equation of the motion of the undamped system is given by

$$y'' + 4y = 2\cos t, \qquad y(0) = y'(0) = 0.$$

Solve the initial value problem and sketch the solution versus $t$.

This type of motion, possessing a periodic variation of amplitude, is called a **beat**.

If $\omega = \omega_0$

$$y_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

Hence, in the *undamped resonant case* $(\omega = \omega_0)$,

$$y(t) = A \sin(\omega_0 t + \phi) + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

$y_p(t)$ oscillates between $-\frac{F_0}{2m\omega_0}$ and $\frac{F_0}{2m\omega_0}$. As $t \to \infty$, the maximum magnitude of $y_p(t) \to \infty$.

**Example 3.** Consider the equation of motion for an undamped system at resonance

$$y'' + 9y = 2\cos(3t), \qquad y(0) = y'(0) = 0$$

Solve the initial value problem and sketch the solution versus $t$.