Section 3.8 **Forced vibrations**

Let's investigate the effect of a cosine forcing function on the system governed by the differential equation

$$mu'' + \gamma u' + ku = F_0 \cos \omega t,$$

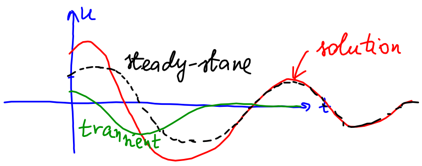
where F_0, γ are nonnegative constants and $\gamma^2 < 4mk$ (the system is underdamped).

The general solution to this equation is

$$u(t) = \underbrace{\boxed{Ae^{-(\gamma/2m)t} \sin\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t + \phi\right)}}_{\text{transient part}} + \underbrace{\boxed{\frac{F_0}{\sqrt{(k-m\omega^2)^2 + \cancel{b}\gamma^2\omega^2}} \sin(\omega t + \theta)}}_{\text{steady-state solution}},$$

where A, ϕ are constants and $\tan\theta = \frac{k-m\gamma}{b\gamma}$.

$$y_h(t) = Ae^{-(\gamma/2m)t} \sin\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t + \phi\right)$$



is called the **transient part** of solution. $y_h \to 0$ as $t \to \infty$.

$$y_p(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + \cancel{b}\gamma^2\omega^2}} \sin(\omega t + \theta)$$

is the offspring of the external forcing function $f(t) = F_0 \cos\omega t$. y_p is sinusoidal with angular frequency ω. y_p is out of phase with $f(t)$ by the angle $\theta - \pi/2$, and its magnitude is different by the factor

$$\frac{1}{\sqrt{(k-m\omega^2)^2 + \cancel{b}\gamma^2\omega^2}}$$

y_p is called the **steady-state** solution.

The factor $\dfrac{1}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}$ is called the **frequency gain** or **gain factor**.

Example 1. A 2-kg mass is attached to a spring with stiffness $k = 45$ N/m. At time $t = 0$, an external force $f(t) = 12\cos 3t$ is applied to the system. The damping constant for the system is 4 N-sec/m. Determine the steady-state solution for the system.

$m = 2, \quad k = 45, \quad \gamma = 4$

$2u'' + 4u' + 45u = 12\cos 3t$

auxiliary equation: $2r^2 + 4r + 45 = 0$

$r_1 = \frac{-4 + \sqrt{16 - 180}}{4} \neq 3$

Form of a particular solution of the equation

$y_p(t) = A\cos 3t + B\sin 3t$

$y_p'(t) = -3A\sin 3t + 3B\cos 3t$

$y_p''(t) = -9A\cos 3t - 9B\sin 3t$

$$2\underbrace{\left(-9A\cos 3t - 9B\sin 3t\right)}_{y_p''} + 4\underbrace{\left(-3A\sin 3t + 3B\cos 3t\right)}_{y_p'} + 45\underbrace{\left(A\cos 3t + B\sin 3t\right)}_{y_p} = 12\cos 3t$$

$$\cos 3t\left(-18A + 12B + 45A\right) + \sin 3t\left(-18B - 12A + 45B\right) = 12\cos 3t$$

$\cos 3t$: $12B + 27A = 12$ or $4B + 9A = 4$

$\sin 3t$: $27B - 12A = 0$ or $9B = 4A$ or $A = \frac{9B}{4}$

$4B + \frac{81B}{4} = 4 \Rightarrow \boxed{B = \frac{16}{97}} \qquad A = \frac{9B}{4} = \frac{16}{97}\cdot\frac{9}{4} = \boxed{\frac{36}{97} = A}$

steady-state solution is $\boxed{y_p(t) = \frac{36}{97}\cos 3t + \frac{16}{97}\sin 3t}$

$y_p(t) = \frac{2\sqrt{373}}{97}\sin(3t + \varphi)$, where $\varphi = \arctan\frac{4}{9}$

gain factor $\frac{2\sqrt{373}}{12(97)} = \frac{\sqrt{373}}{582}$

In general, the amplitude of the steady-state solution depends on ω and is given by

$$A(\omega) = F_0 M(\omega) = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

$M(\omega)$ is the frequency gain. The graph of $M(\omega)$ is called the **frequency response curve** or **resonance curve** for the system.

$M'(\omega) = 0$ when either $\omega = 0$ or

$$\boxed{\omega = \omega_r = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}}$$

resonance frequency for the system.

When the system is critically damped or overdamped, $M'(\omega) = 0$ only when $\omega = 0$. When $\gamma^2 - 2mk < 0$, then $M'(\omega) = 0$ at

$$\omega_r = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}$$

The value $\omega_r/2\pi$ is called the **resonance frequency** for the system. When the system is stimulated by an external force as this frequency, it is said to be **at resonance**.

Consider the undamped system ($\gamma = 0$) with forcing term $F_0 \cos \omega t$. This system is governed by

$$m \frac{d^2 y}{dt^2} + ky = F_0 \cos \omega t$$

$$y(t) = y_h(t) + y_p(t)$$

$$y_h(t) = A \sin(\omega_0 t + \phi), \qquad \omega_0 = \sqrt{k/m}$$

If $\omega \neq \omega_0$

$$y_p(t) = \frac{F_0}{k - m\omega^2} \sin(\omega t + \theta)$$

$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$

Example 2. The equation of the motion of the undamped system is given by

$$y'' + 4y = 2\cos t, \qquad y(0) = y'(0) = 0.$$

Solve the initial value problem and sketch the solution versus t.

$y''+4y=0$, auxiliary equation $r^2+4=0$, $r=\pm 2i$

$y_h(t) = c_1 \cos 2t + c_2 \sin 2t$ trannient part

steady-state solution $y_p(t) = A\cos t + B \sin t$

$A = \frac{2}{3}$, $B=0$.

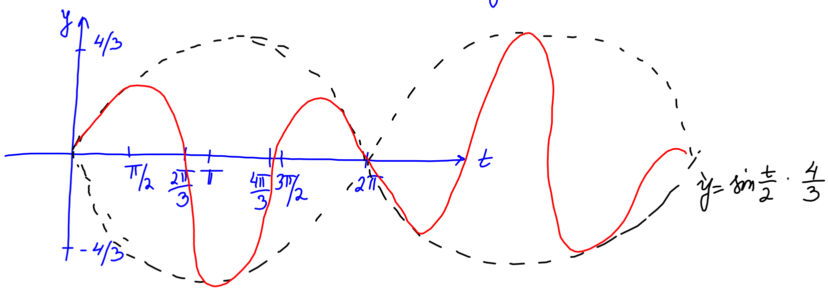
General solution is $y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{2}{3}\cos t$

$y(0) = c_1 + \frac{2}{3} = 0 \Rightarrow c_1 = -\frac{2}{3}$

$y'(0) = 2c_2 = 0 \Rightarrow c_2 = 0$

$y(t) = -\frac{2}{3}\cos 2t + \frac{2}{3}\cos t = -\frac{2}{3}(-2)\sin\frac{2t+t}{2}\sin\frac{2t-t}{2}$

$y(t) = \frac{4}{3}\sin\frac{3t}{2}\sin\frac{t}{2}$



If $\omega = \omega_0$

$$y_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

Hence, in the *undamped resonant case* ($\omega = \omega_0$),

$$y(t) = A \sin(\omega_0 t + \phi) + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

$y_p(t)$ oscillates between $-\dfrac{F_0}{2m\omega_0}$ and $\dfrac{F_0}{2m\omega_0}$. As $t \to \infty$, the maximum magnitude of $y_p(t) \to \infty$.

Example 3. Consider the equation of motion for an undamped system at resonance

$$y'' + 9y = 2\cos(3t), \qquad y(0) = y'(0) = 0$$

Solve the initial value problem and sketch the solution versus t.

$y_h(t) = c_1 \cos 3t + c_2 \sin 3t$ trannient part

steady-state solution $y_p(t) = (A \cos 3t + B \sin 3t)t$

$B = \frac{1}{3}$, $A = 0$

$y(t) = C_1 \cos 3t + C_2 \sin 3t + \frac{1}{3} t \sin 3t$

$y(0) = C_1 = 0$

$y'(0) = 3C_2 = 0$

$y(t) = \frac{1}{3} t \sin 3t \longrightarrow \infty$ as $t \to \infty$